DISCRETE-TIME MARTINGALES AND OPTIONAL STOPPING THEOREM

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1. Introduction to Martingales

Definition 1.1. A *martingale* is a sequence of random variables with finite mean $X_1, X_2, \ldots, X_n$, such that $\mathbb{E}(X_n | X_{n-1} \ldots X_1) = X_{n-1}$.

In some instances it is required that X_n be a function of some other sequence $M_0, \ldots, M_n$.

Definition 1.2. Given some martingale X and the random variable X_n defined as the position of the martingale at time n, the *filtration* $\mathcal{F}_n$ is the set $\{X_1, \ldots, X_{n-1}\}$.

A *filtration* $\mathcal{F}$ has a rather involved definition, but for this paper, it is defined conditionally. In fact, for the rest of the paper, we eschew most measure theory.

Hence, Definition 1.1 can be revised more succinctly to be a set of random variables $X_1, \ldots, X_n$ s.t. $\mathbb{E}(X_n | \mathcal{F}_n) = X_{n-1}$ for all $n \geq 1$.

Definition 1.3. A *predictable sequence* is a sequence of random variables Z such that Z_n depends only on $X_1 \ldots X_{n-1}$ and not on some future value. That is, $Z_n = f(X, \ldots, X_{n-1})$.

Definition 1.4 (Martingale Transforms)**.** Let $\gamma_k = X_k - X_{k-1}$ be a *martingale difference* and let $\{Z_n\}_{(n \geq 1)}$ be a predictable sequence. A *martingale transform* is defined

$$(Z \cdot X) = X_0 + \sum_{i=1}^{n} Z_i \gamma_i.$$

Example. A gambler flips a coin. If it turns up heads, let $\gamma_i = 1$. If it is tails, let $\gamma_i = -1$. Depending on the outcome of the previous throws, assign a bet that the next flip is heads. Then, it is clear that the sequence of bets, say Z, is a predictable sequence (since it depends only on $\mathcal{F}_n$). Similarly, define $X_n = \gamma_1 + \ldots \gamma_n$. Since $\mathbb{E}(\gamma_i) = 0$, $\mathbb{E}(X_n) = \mathbb{E}(X_{n-1} + \gamma_n) = \mathbb{E}(X_{n-1}) = X_{n-1}$, so X is a martingale.

Our Martingale Transform $(Z \cdot X)_n$ represents the amount of money the gambler has at the nth flip.

Example. Instead of coin flips, consider the more general bernoulli trial, or really, any probability distribution Y s.t. $\mathbb{E}(Y) = 0$. By similar logic to above, we know that $X_n = Y_1 + \ldots Y_n$ is a martingale. We can determine some function that assigns a bet on $(Z \cdot X)_n$ depending on the number of successes in $\mathcal{F}_n$ (identical to above).

Martingale Transforms are immensely useful constructs. Since they depend only on $\mathcal{F}_{n-1}$, it makes intuitive sense that Transforms may also be martingales themselves. This intuition turns out to be true.

Lemma 1.5. *The martingale transform $(Z \cdot X)_n$ (see Definition 1.4) is also a martingale.*

Proof. We wish to show $\mathbb{E}((Z \cdot X)_n) = (Z \cdot X)_{n-1}$. Recall $Z \cdot X = X_0 + \sum_{i=1}^{n} Z_i\gamma_i = X_0 + \sum_{i=1}^{n-1} Z_i\gamma_i + Z_n\gamma_n$. By linearity of expectation, $\mathbb{E}((Z \cdot X)_n) = \mathbb{E}(X_0 + \sum_{i=1}^{n-1} Z_i\gamma_i) + \mathbb{E}(Z_n \cdot \gamma_n) = \mathbb{E}((Z \cdot X)_{n-1}) + \mathbb{E}(Z_n\gamma_n)$. If $\{Z\}$ and $\{\gamma\}$ are independent, $\mathbb{E}(Z_n\gamma_n) = \mathbb{E}(Z_n)\mathbb{E}(\gamma_n)$.

Since, $\mathbb{E}(X_k) = X_{k-1} = \mathbb{E}(X_{k-1})$, by Linearity of Expectation, we have $\mathbb{E}(X_k - X_{k-1}) = \mathbb{E}(\gamma_k) = 0$.

Thus, $\mathbb{E}((Z \cdot X)_n) = \mathbb{E}((Z \cdot X)_{n-1}) + \mathbb{E}(Z_n\gamma_n) = \mathbb{E}((Z \cdot X)_{n-1}) = (Z \cdot X)_{n-1}$. ■

2. Optional Sampling Theorem

Although generally used as an intermediate step to ultimately prove Optional Stopping Theorem, Doob's Optional Sampling Theorem is intersting in its own right. In particular, it demonstrates the extreme power of Martingale Transforms.

Definition 2.1. A *stopping time* relative to a filtration $\{\mathcal{F}_n\}_{\geq 0}$ is a non-negative integer-valued random variable τ such that for each n the event $\{\tau = n\} \in \mathcal{F}_n$. That is, the stopping time is determined the information up to and including n.

Example. Consider a lazy random walk of an ant on the number line. The ant stops when it reaches either $x = -2$ or $x = 2$ for the first time.

Nonexample. Consider a lazy random walk of an ant on the number line. The last time the ant reaches $x = 2$ is *not* a stopping time. This is because it relies on information that will come in the future (i.e. is not included in $\mathcal{F}_n$).

Theorem 2.2. *Let $a \wedge b$ refer to min(a, b). The stopped sequence $\{X_{n\wedge\tau}\}_{n\geq 0}$ is a martingale.*

Proof. Since a stopping time relies only on $\mathcal{F}_n$, it seems likely that a stopped martingale is also a Martingale Transform. To confirm this, we need to explicitly construct our predictable sequence $\{Z_n\}_{(n\geq 1)}$.

We claim the following construction for $\{Z_n\}_{(n\geq 1)}$ holds:

$$\begin{cases} 1 & \tau \geq n \\ 0 & \tau < n \end{cases}$$

Verifying, note that $(Z \cdot X) = X_0 + \sum_{i=1}^{n} Z_i\gamma_i = X_0 + \sum_{i=1}^{n\wedge\tau} \gamma_i = X_{\tau\wedge n}$. ■

3. Optional Stopping Theorem

By Optional Sampling Theorem, we know that $X_{n\wedge\tau} = X'$ is also a martingale. As n approaches infinity, the martingale approaches X_τ, which, by Optional Sampling Theorem, must be a martingale. In particular, we need to find conditions such that

$$\lim_{n\to\infty} \mathbb{E}(X') = \mathbb{E}(X_\tau).$$

Then, it is an obvious corollary that $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$. That is, we know (X') is a martingale by Optional Sampling Theorem, so $\mathbb{E}(X') = \mathbb{E}(X_0) = 0$. Therefore, if $\lim_{n\to\infty} \mathbb{E}(X') = \mathbb{E}(X_\tau)$, then the result follows.

Definition 3.1. A martingale X_n is *uniformly integrable* (UI) if $\sup_{n\geq 0}(\mathbb{E}(|X_n| I\{|X_n| > x\}))$ approaches 0 as x approaches infinity.

Proposition 3.2. *If a martingale X is UI, then* $\lim_{n\to\infty} \mathbb{E}(X') = \mathbb{E}(X_\tau)$*, implying* $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$.

Armed with this knowledge, we can begin to prove the main result, the *Optional Stopping Theorem.*

Lemma 3.3 (Dominated Convergence Theorem)**.** *If X_n approaches X as n approaches ∞ and $\sup|X_n| \leq Y$ for some random variable Y with $\mathbb{E}(Y) < \infty$, then $\mathbb{E}(X_n)$ approaches $\mathbb{E}(X)$ as n approaches ∞.*

For the sake of notation, this can also be written as: If $X_n \to X$, then $\mathbb{E}(X_n) \to \mathbb{E}(X)$. Before we can prove this statement, we need to prove another Lemma.

Lemma 3.4 (Fatou's Lemma)**.** *If $X_1, X_2, \ldots$ are non-negative random variables and $X_n \to X$, then $\mathbb{E}\lim_{n\to\infty}\inf X_n \leq \lim_{n\to\infty}\inf \mathbb{E}X_n$.*

Proof. Define a new sequence $Y_n = \inf_{k\geq n} X_k$. This is a non-decreasing sequence which converges to $\lim_{n\to\infty}\inf \mathbb{E}X_n$. Since $X_n \geq Y_n$, $\lim_{n\to\infty}\inf \mathbb{E}X_n \geq \lim_{n\to\infty}\inf \mathbb{E}Y_n = \lim_{n\to\infty}\mathbb{E}(Y_n)$ since Y_n is non-decreasing and convergent. By applying monotone convergence theorem (not proved in the paper), we have $\lim_{n\to\infty}\mathbb{E}(Y_n) = \mathbb{E}(\lim_{n\to\infty} Y_n) = \mathbb{E}\lim_{n\to\infty}\inf X_n$. ■

Now, we finish the proof of Dominated Convergence Theorem.

Proof of Lemma 3.3. Recall $|X_n| \leq Y$, so $|X| \leq Y$ as $n \to \infty$. This implies, $|X - X_n| \leq 2Y$. Applying Fatou's Lemma, we have:

$$\mathbb{E}(2Y) = \mathbb{E}\lim_{n\to\infty}\inf(2Y - |X_n - X|) \leq \lim_{n\to\infty}\inf\mathbb{E}(2Y - |X_n - X|) = 2\mathbb{E}Y - \lim_{n\to\infty}\sup\mathbb{E}(|X_n - X|).$$

Therefore, $\lim_{n\to\infty}\sup\mathbb{E}(|X_n - X|) \leq 0$, implying $\mathbb{E}|X_n - X| \to 0$. Trivially, we have $|\mathbb{E}X_n - X| \leq \mathbb{E}|X_n - X| \to 0$, so $|\mathbb{E}X_n - X| \to 0$, implying $\mathbb{E}X_n \to \mathbb{E}X$.

■

As a trivial corollary, if a martingale satisfies Dominated Convergence Theorem (henceforth referred to as DCT), then it is UI.

Theorem 3.5 (Optional Stopping Theorem)**.** *Each of the following conditions are equivalent and must hold in order for $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$.*

1. $\sup_{n\geq 0}|X'| \leq Y$ *where Y is a r.v. such that $\mathbb{E}(Y)$ is finitely bounded.*
2. *The stopping time τ is bounded.*
3. $\mathbb{E}(|X_\tau|) < \infty$ *and* $\mathbb{E}(|X_\tau|; \tau > n)$ *approaches* 0 *as n approaches ∞.*

Proof of Theorem 3.5. It was shown that Optional Stopping Theorem is a corollary of Optional Sampling Theorem as long as the Martingale is UI. Thus, it is enough to show that the conditions satisfy UI.

1. This is the direct statement of Dominated Convergence. As previously asserted (but not proved), any martingale that satisfies DCT is UI.
2. This is a corollary of (1). In particular, set the random variable Y to be $\max\{|X_1|, \ldots, |X_n|\}$. Then, DCT applies again.
3. Using a technique similar to the proof or Optional Sampling Theorem, we split $|X'|$ into $|X_\tau|I\{\tau < n\} + |X_n|I\{\tau \leq n\}$. We know $\mathbb{E}(|X_\tau|; \tau > n)$ approaches 0 as n approaches ∞, so

$$\lim_{n\to\infty}\mathbb{E}(|X'|) = \lim_{n\to\infty}\mathbb{E}(|X_\tau|I\{\tau \leq n\}).$$

Furthermore, it is obvious that $|X_\tau| I\{\tau \leq n\} \leq |X_\tau$ and $\mathbb{E}(|X_\tau| < \infty$ is true by assumption. Now, DCT applies, with the random variable Y as X_τ.

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4. Random Walks

Martingales, and especially Optional Stopping Theorem, have numerous applications. Perhaps most importantly, it significantly reduces the amount of time necessary to compute various facts about the symmetric random walk.

Definition 4.1. A *simple symmetric random walk* of size n is defined as $\sum_{i=0}^{n} Z_i$ where Z_i is a random variable that is 1 with probability $1/2$ and -1 with probability $1/2$.

Lemma 4.2. *The symmetric random walk is a martingale.*

Proof. Let X_i refer to the position of the simple random walk at time i. Now, note by Definition 3.1 that $\mathbb{E}(X_{i+1}) = \frac{1}{2}(X_i - 1) + \frac{1}{2}(X_i + 1) = X_i$. Therefore, $\mathbb{E}(X_{i+1}) = X_i$, implying $\mathbb{E}(X_{i+1}|\mathcal{F}_n) = X_i$. ■

Theorem 4.3. *For a random walk beginning at $x = 0$ and ending when the walker first reaches $x = -a$ or $x = b$, the probability of hitting $x = a$ first is $\frac{b}{a+b}$.*

Proof. Since the simple symmetric random walk is a martingale, we may freely use the Optional Stopping Theorem. Our stopping time is defined as $\tau = a \wedge b$ where a is the position $x = a$ and b is the position $x = b$. Recalling Theorem 3.5, we have $\mathbb{E}(X_\tau) = \mathbb{E}(X_0) = 0$. Let p_1 be the probability of stopping at $-a$ and let p_2 be the probability of stopping at b. Note, we can rewrite $\mathbb{E}(X_\tau)$ as $p_1 \cdot (-a) + p_2 \cdot (b) = 0$. Clearly, by the constraints of Optional Stopping Theorem, $p_1 + p_2 = 1$. Hence, we have a system of linear equations:

$$p_1 \cdot -a + p_2 \cdot b = 0$$

$$p_1 + p_2 = 1.$$

Solving the system, we have $p_1 = \frac{b}{a+b}$ and $p_2 = \frac{a}{a+b}$. ■

Theorem 4.4. *For a random walk beginning at $x = 0$ and ending when the walker first reaches $x = -a$ or $x = b$, the expected number of moves until the walk ends is ab.*

First, we prove a lemma.

Lemma 4.5. *If the sequence $\{X_n\}$ is a martingale representing a simple random walk, then the sequence $\{X_n^2 - n\}$ is also a martingale.*

Proof. Define a new martingale $\{M_n\}$ as $\{X_n^2 - n\}$. We wish to show $\mathbb{E}(M_n) = M_{n-1}$ ir $\mathbb{E}(X_n^2 - n) = X_{n-1}^2 - n$. Since this is a random walk, note that $X_n = Z_1 + \ldots Z_n$ where Z_i is ± 1. So, $\mathbb{E}(X_n^2 - n) = \mathbb{E}(X_{n-1} + Z_n)^2 - n = \mathbb{E}(X_{n-1}^2 + 2X_{n-1}Z_n + Z_n^2) - n$. Applying linearity of expectation, we have $\mathbb{E}(X_n^2 - n) = \mathbb{E}(X_{n-1}^2) + 2\mathbb{E}(X_{n-1}Z_n) + 1 - n = \mathbb{E}(X_{n-1}^2 - (n-1))$, which finishes the proof. Note, in this case $E(X_{n-1}^2 - (n-1)) = X_{n-1}^2 - (n-1)$ since the value of X_{n-1} is already known. ■

Now, with a bit of algebra, we can finish the Theorem.

Proof of Theorem 4.4. Since the previously defined M_n is a martingale, we can apply Optional Stopping Theorem. In particular $\mathbb{E}(M_\tau) = \mathbb{E}(M_0) = 0$. Relabeling the stopping time as n, we have $\mathbb{E}(X_n^2 - n) = 0$, and by linearity of expectation, $\mathbb{E}(X_n^2) = \mathbb{E}(n)$. Now, we explicitly compute $\mathbb{E}(X_n^2)$. With probability p_1 we end at $x = -a$ and with probability p_2 we end at $x = b$. So, the expected value of $X_\tau^2 = p1(a^2) + p_2(b^2)$. Recall that $p_1 = \frac{b}{a+b}$ and $p_2 = \frac{a}{a+b}$, so $\mathbb{E}(n) = \frac{ab^2+ba^2}{a+b} = ab$. ■

References

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