MARTINGALES

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Abstract. A martingale, put simply, is just a variable that at any time, the expected value of that variable at the next time iteration is equal to its present value, no matter its path to get to the present. Martingales originated in the saloons of 18th century France, where people tried to find better betting strategies for things like dice rolls, and coin flips. Over the years, martingale theory has become quite sophisticated, with applications from simple walks on a line to even election theory. In this article, we provide a brief introduction and definition of martingale theory and martingales themselves, along with a few simple applications in context.

1. Introduction and Definitions

As mentioned above, the rough definition of a martingale is simply a process on a probability space that is conditionally expected to stay constant at any point in time regardless of previous values. It's a stochastic process, say M_t, on some space say, $(\Omega, \mathcal{F}, P)$, which we define to be indexed by some set $\mathcal{T}$. We usually define the index set T such that all times $t \in \mathcal{T}$, and $\mathcal{T}$ is often the positive integers or positive reals. Let M_T be our martingale, and for our conditional expectation to be well-defined we require

$$E(M_t) < \infty$$

for all t. The information that is known at a time t we represent as a family of increasing σ-algebras $\{\mathcal{F}_t\} \subset \mathcal{F}$. In addition to the finite requirement, we also need $M_t = E[M_s|\mathcal{F}_t]$ where $t < s$. For things like Markov chains, we only need to consider $s = t + 1$, since they are integer time processes, and take $\mathcal{F}_t = \sigma[X_i : i \leq t]$ and write out process as:

$$M_t = E[M_{t+1}|X_0, X_1, ...X_t]$$

This concludes our basic definitions sections, the rest of this article will deal with a key theorem about integer-timed martingales.

2. Optimal Stopping Time Theorem

One of the key theorems in martingale theory is the optimal stopping time theorem. Lets define a random time $\tau \in T$ to be a "Markov time" if τ doesn't depend on the future. More formally, if

$$[\tau \leq t] \in \mathcal{F}_t$$

for each t in $\mathcal{T}$. If τ is a stopping time and M_τ is a martingale, then $M_{t\wedge\tau}$ is also a martingale too, by the theorem. We can prove this quite easily: our finite condition is easily seen to be satisfied. Now let's look at $E[M_{(t+1)\wedge\tau}|\mathcal{F}_t]$, taking $s = t + 1$ as explained before, for integer-timed martingales. We can separate this into:

$$E[M_{(t+1)\wedge\tau}|\mathcal{F}_t] = E[M_\tau \mathbb{1}_{[\tau\leq t]} + M_{t+1}\mathbb{1}_{[\tau\geq t]}|\mathcal{F}_t]$$

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By the definition of M_t being a martingale, $M_t = E[M_s|\mathcal{F}_t]$, and so we can simplify the above into:

$$E[M_\tau \mathbb{1}_{[\tau\leq t]} + M_{t+1}\mathbb{1}_{[\tau\geq t]}|\mathcal{F}_t] = M_\tau \mathbb{1}_{[\tau\leq t]} + \mathbb{1}_{[\tau\geq t]}E[M_{t+1}|\mathcal{F}_t] = M_\tau \mathbb{1}_{[\tau\leq t]} + \mathbb{1}_{[\tau\geq t]}M_t$$

This is just:

$$M_\tau \mathbb{1}_{[\tau\leq t]} + \mathbb{1}_{[\tau\geq t]}M_t = M_{t\wedge\tau}$$

and thus we've shown that $M_{t\wedge\tau}$ is martingale, and this proves the optimal stopping theorem.

3. A Simple Random Walk

Let $\{\beta_j\}$ be random independent variables with $\beta_j \in \{-1, 1\}$. For a fixed $0 < p < 1$ let's have $P[\beta_j = 1] = p, P[\beta_j = -1] = q = 1 - p$. We also set $\mathcal{F}_n = \sigma\{\beta_j : j \leq n\}$. Let $x \in \mathbb{Z}$, and define

$$X_n = x + \sum_{j\leq n} \beta_j$$

We've just defined a simple random walk, and we denote one to be symmetric if $p = q = \frac{1}{2}$. Consider the following three processes:

$$M_n^{(1)} = X_n - \alpha n$$

$$M_n^{(2)} = (q/p)^{X_n}$$

$$M_n^{(3)} = (X_n - \alpha n)^2 - 4pqn$$

Where again, $q = 1 - p$ and $\alpha = p - q$. For clarity, we consider these processes over $n \in \mathbb{Z}_+$. Each of these is clearly a martingale, which the reader can verify by simple induction using the total expectation rule to separate them. Let's use these to investigate when a simple random walk like this would move around in some bounds. More specifically, if we take two integers a, b such that $a \leq x$ and $b \geq x$, we want to find out how often the walk leaves these bounds say to the right, exiting at b. This can be used to model many things, from elections to even roulette. Let's begin setting out stopping time $\tau := \inf\{t \geq 0 : X_t \notin (a, b)\}$. Let

$$K = [X_\tau = b] \cap [\tau < \infty]$$

denote the event that we want, namely our walk exits at b before it exits at a. Now, let's first consider a symmetric walk. In that event, we know that $E[M_0^{(1)}] = x$. By the Dominated Convergence Theorem,

$$E[M_0^{(1)}] = x = \lim_{t\longrightarrow\infty} E[M_{t\wedge\tau}^{(1)}$$

$$\lim_{t\longrightarrow\infty} E[M_{t\wedge\tau}^{(1)} = E[M_\tau^{(1)}] = bP[K] + a(1 - P[K])$$

Now, just solving for $P[K]$ yields

$$P[K] = \frac{x - a}{b - a}$$

which is quite elegant and concise. Let's tackle the case where the walk is not symmetric. We'll use the second martingale now. Since $p \neq q$, $(\frac{p}{q})^b \neq (\frac{p}{q})^a$. We can do a similar thing as before with the Dominated Convergence Theorem:

$$\begin{aligned} E[M_0^{(2)}] &= (\frac{q}{p})^x \\ &= \lim_{t \longrightarrow \infty} E[M_{t\wedge\tau}^{(2)}] = E[M_\tau^{(3)}] \\ &= P[K](\frac{q}{p})^b + (1 - P[K])(\frac{q}{p})^a \end{aligned}$$

Some simple algebra solving or $P[K]$ should yield the following: t

$$P[K] = \frac{(q/p)^x - (q/p)^a}{(q/p)^b - (q/p)^a}$$

Let's say we want to find out how long the walk lasts between b, a. For the symmetric case, we use the third martingale, as when $p = q = 1/2, 4pq = 1$. Well,

$$\begin{aligned} M_0^{(3)} = x^2 &= \lim_{t \longrightarrow \infty} E[M_{t\wedge\tau}^{(2)}] = E[M_\tau^{(2)}] \\ &= E(X_\tau^2 - \tau] = P[K]b^2 + (1 - P[K])a^2 - E[\tau] \\ &= \frac{a^2(b-x) + b^2(x-a)}{b-a} - E[\tau] \end{aligned}$$

Solving for for $E[\tau]$ yields

$$E[\tau] = (a+b)x - x^2 - ab = (x-a)(b-x)$$

For example, if we set $a = 0, b = 50$, and start our walk at $x = 25$, then the expected number of steps it would take to exit would be $(25)(25) = 625$ moves. We leave the not symmetric case as an exercise for the reader, the derivation is much the same, consider $x = M_0^{(1)} = E[M_\tau^{(1)}$ and proceed with the calculation.

There's some cool application of these results, for example in the popular casino game roulette, in which a 1:1 bet yields a $p = 9/19$ chance of winning the round. It's startling the difference the 1/19 difference from symmetric can make, the calculations yield that with 1 dollar bets, and starting at say $x = 50$, with $a = 0, b = 100$ yields that $P[K] = 0.0051$, while in a symmetric game this would be 0.5.

References

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