ALDOUS'S SPECTRAL GAP CONJECTURE

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1. Introduction

The spectral gap of a Markov chain is related to the eigenvalues of the transition matrix, and can help describe how fast that Markov chain converges to the stationary distribution. A version of Aldous's Spectral Gap Conjecture was originally stated around 1992, and various special cases of Aldous's Spectral Gap Conjecture have been proven since. It was finally proved in 2010 by Caputo, Liggett, and Richthammer. This conjecture states that the spectral gap of the random walk on a graph is equal to the spectral gap of the interchange process on that graph. In this paper, we introduce background related to the spectral gap and interchange process, and prove the conjecture. We assume basic knowledge of Markov chains, graph theory, and linear algebra.

2. Spectral Gap of a Random Walk

Let $Z = (Z_i)_{i\geq 0}$ be a Markov chain with finite state space S and transition probabilities q_{ij}. Assume that the Markov chain is irreducible and that the transition matrix is symmetric, i.e. $q_{ij} = q_{ji}$. We first define the spectral gap:

Definition 2.1. The **spectral gap** λ_1 is the smallest positive eigenvalue of $-Q$, where Q is the transition matrix.

We can interpret the spectral gap in terms of the mixing time:

Definition 2.2. We define the **relaxation time** t_{rel} as $\frac{1}{\lambda^*}$, where λ^* is the absolute spectral gap.

Theorem 2.3. *Let $\pi_{\min} = \min_{x\in\Omega} \pi_x$. Then,*

$$(t_{\text{rel}} - 1)\log\frac{1}{2\varepsilon} \leq t_{\text{mix}}(\varepsilon) \leq \log\left(\frac{1}{\varepsilon\pi_{\min}}\right) t_{\text{rel}}$$

Corollary 2.4. *As $t \to \infty$,*

$$\mathbb{P}_i(Z_t = j) = \pi_j + a_{ij}e^{-\lambda_1 t} + o(e^{-\lambda_1 t})$$

where a_{ij} is a constant depending on i and j.

We can also interpret the spectral gap in terms of the infinitesimal generator, which we define below:

Definition 2.5. The **infinitesimal generator** of a Markov chain with transition matrix Q is the linear operator

$$\mathcal{L}g(i) = \sum_{j\in S} q_{i,j}(g(j) - g(i))$$

where the input g is a function from $S \to \mathbb{R}$, and $i \in S$.

Proposition 2.6. *The spectral gap λ_1 is the largest constant λ such that*

$$\frac{1}{2}\sum_{i,j\in S} q_{ij}(g(j)-g(i))^2 \geq \lambda \sum_{i\in S} g(i)^2$$

for all functions $g : S \to \mathbb{R}$ with $\sum_i g(i) = 0$.

Equivalently, we have the following:

Corollary 2.7. *The spectral gap λ_1 is the largest constant λ such that*

$$\sum_{z\in S}(\mathcal{L}g(z))^2 \geq -\lambda \sum_{z\in S} g(z)\mathcal{L}g(z)$$

A consequence of this is that $\mathcal{L}^2 + \lambda\mathcal{L}$ is nonnegative definite, i.e. its eigenvalues are nonnegative.

3. The Interchange Process

Let $G = (V, E)$ be a weighted undirected graph with edge weights c_{xy}.

Definition 3.1. In the **interchange process**, we assign a labeling to the vertices of the graph and jump from state η to state η^{xy} with probability c_{xy}, transitioning by switching the labels at vertices x and y. Let $\mathcal{X}_n$ denote the permutations of $\{1, 2, \dots, n\}$, and for $\eta \in \mathcal{X}_n$ and $xy \in E$, we define $\eta^{xy} = \eta\tau_{xy}$ where $\tau_{xy} \in \mathcal{X}_n$ is the transposition $(x\ y)$.

Similar to the way we defined infinitesimal generators of the random walk, we can define it in the same way for the interchange process:

Definition 3.2. The infinitesimal generator of the interchange process is the linear operator

$$\mathcal{L}^{IP} f(\eta) = \sum_{xy\in E} c_{xy}(f(\eta^{xy}) - f(\eta))$$

where the input is a function $f : \mathcal{X}_n \to \mathbb{R}$, and η is an element of $\mathcal{X}_n$.

The linear operator can be represented in terms of a matrix, which allows us to define the spectral gap for the interchange process:

Definition 3.3. The spectral gap λ_1^{IP} is the smallest positive eigenvalue of $-M$, where M is the matrix representing $\mathcal{L}$.

4. Special Cases of Aldous' Spectral Gap Conjecture

In this section, we will look at special cases or weaker forms of Aldous's Spectral Gap Conjecture. We first show the following:

Proposition 4.1. *For all weighted graphs G, we have $\lambda_1^{IP}(G) \leq \lambda_1^{RW}(G)$.*

Proof. We use the proof from [CLR10]. The idea of the proof is that the random walk is a subprocess of the interchange process, because we get a random walk if we just focus on the vertex with label 1. Let $\mathcal{L}_1$ represent the infinitesimal generator of the random walk, and let $\mathcal{L}_2$ represent the infinitesimal generator of the interchange process. We denote the state space of the random walk as S_1 and the state space of the interchange process as S_2.

Since the random walk is a subprocess of the interchange process, this gives a surjective function $f : S_2 \to S_1$ such that $\mathcal{L}_2(g \circ f) = (\mathcal{L}_1 g) \circ f$ for all functions $g : S_1 \to \mathbb{R}$. We want to show that if g is an eigenfunction of $-\mathcal{L}_1$, then $f \circ g$ is an eigenfunction of $-\mathcal{L}_2$. If $-\mathcal{L}_1 g = \lambda g$, then we have

$$-\mathcal{L}_2(g \circ f) = (-\mathcal{L}_1 f) \circ \pi = \lambda g \circ f$$

which shows that $\operatorname{Spec}(-\mathcal{L}_1) \subset \operatorname{Spec}(-\mathcal{L}_2)$. Thus, $\lambda_1^{IP} \leq \lambda_1^{RW}$, as desired. □

In 1980, Diaconis and Shahshahani showed that for an unweighted complete graph (for which we can just assign each edge a weight of 1), the spectral gap of the random walk is actually equal to the spectral gap of the interchange process. Note that in this case, we can think of the interchange process as shuffling a stack of n cards, where we pick two positions at random and switch the cards at those positions. We then ask how many transpositions should be made for the permutation to be random, i.e. the distribution is close to the stationary distribution.

Proposition 4.2 (Diaconis–Shahshahani)**.** *For a complete graph on n vertices where $c_e = 1$ for all $e \in E$, we have $\lambda_1^{IP} = \lambda_1^{RW}$.*

Proof. We refer the reader to [DS81] for an in-depth proof and discussion. Since the interchange process is related to the permutation group S_n, this paper uses methods from group theory to compute the eigenvalues of these processes. □

Using similar methods, Flatto, Odlyzko, and Wales showed the following in 1985:

Proposition 4.3 (Flatto–Odlyzko–Wales)**.** *For a star graph where $c_e = 1$ for all $e \in E$, we have $\lambda_1^{IP} = \lambda_1^{RW}$.*

Proof. See [FOW85]. □

Cesi further used these methods in 2009 to extend the results for multipartite graphs:

Proposition 4.4 (Cesi)**.** *For complete multipartite graphs where $c_e = 1$ for all $e \in E$, we have $\lambda_1^{IP} = \lambda_1^{RW}$.*

Proof. See [Ces10]. □

It turns out that this result holds for a general graph. This is Aldous's Spectral Gap Conjecture, which was proven in 2010 by Caputo, Liggett, and Richthammer:

Theorem 4.5. *For all weighted graphs G, the interchange process and the random walk have the same spectral gap, i.e.*

$$\lambda_1^{IP}(G) = \lambda_1^{RW}(G).$$

5. Key Results for the Proof of the Conjecture

We present the proof from [CLR10], which uses induction on the number of vertices in the graph. We first define the following:

Definition 5.1. Given a weighted graph $G = (V, E)$ and a point $x \in V$, define the **reduced network** G_x as the graph obtained when removing x from G, i.e. $V_x = V \setminus x$, $E_x = \{yz \in E, y, z \neq x\}$, and the edge weights are

$$c'_{yz} = c_{yz} + c_{yz}^{*,x},\ c_{yz}^{*,x} = \frac{c_{xy} c_{yz}}{\sum_{w \in V_x} c_{xw}}$$

Note that the infinitesimal generator for the random walk on G_x is the same as it is on G, except with modified edge weights:

$$\mathcal{L}_x g(z) = \sum_{y \in V_x} c'_{zy}(g(y) - g(z))$$

for $g : V \to \mathbb{R}$ and $x \in V$.

Proposition 5.2. *For a weighted graph G and a point x, we have*

$$\lambda_1^{RW}(G_x) \geq \lambda_1^{RW}(G).$$

We first prove a lemma:

Lemma 5.3.

$$Lg(z) = \begin{cases} L_x g(z) & z \in V_x \\ 0 & z = x \end{cases}$$

Proof. Let $x \in V$ be arbitrary. Let L represent the infinitesimal generator of the random walk on G, and let L_x represent the infinitesimal generator of the random walk on G_x. Take $g : V \to \mathbb{R}$ such that $Lg(x) = 0$. Then,

$$\sum_{y \neq x} c_{xy}(g(y) - g(x)) = 0$$

We solve for $g(x)$:

$$\sum_{y \neq x} c_{xy} g(y) = g(x) \sum_{w \neq x} c_{xw}$$

$$g(x) = \frac{\sum_{y \in V_x} c_{xy} g(y)}{\sum_{w \in V_x} c_{xw}} \tag{1}$$

Then for $z \in V_x$,

$$Lg(z) = \sum_{y \in V_x} c_{zy}(g(y) - g(z)) + c_{zx}(g(x) - g(z))$$

where we split the separate out the x term. Using 1,

$$Lg(z) = \sum_{y \in V_x} c_{zy}(g(y) - g(z)) + c_{zx}\left(\frac{\sum_{y \in V_x} c_{xy} g(y)}{\sum_{w \in V_x} c_{xw}} - g(z)\right)$$

$$= \sum_{y \in V_x} c_{zy}(g(y) - g(z)) + c_{zx}\left(\frac{\sum_{y \in V_x} c_{xy} g(y) - \sum_{w \in V_x} c_{xw} g(z)}{\sum_{w \in V_x} c_{xw}}\right)$$

$$= \sum_{y \in V_x} c_{zy}(g(y) - g(z)) + \frac{c_{zx}}{\sum_{w \in V_x} c_{xw}}\left(\sum_{y \in V_x} c_{xy}(g(y) - g(z))\right)$$

$$= \sum_{y \in V_x} c_{zy}(g(y) - g(z)) + \frac{c_{xy} c_{xz}}{\sum_{w \in V_x} c_{xw}}\left(\sum_{y \in V_x} (g(y) - g(z))\right)$$

$$= \sum_{y \in V_x} (c_{zy} + c_{zy}^{*,x}(g(y) - g(z)) = \sum_{y \in V_x} c'_{zy}(g(y) - g(z)) = L_x g(z)$$

We have shown that

$$Lg(z) = \begin{cases} L_x g(z) & z \in V_x \\ 0 & z = x \end{cases}$$

as desired. □

Proof of Proposition 5.2. Recall that the spectral gap λ of a process with infinitesimal generator $\mathcal{L}$ is the largest constant satisfying

$$\sum_{z \in V} (\mathcal{L}g(z))^2 \geq -\lambda \sum_{z \in V} g(z)\mathcal{L}g(z).$$

We want to show that $\lambda_1^{RW}(G)$, the spectral gap for the random walk on G, satisfies this inequality for L_x. By Lemma 5.3, we have

$$\sum_{z \in V_x} (L_x g(z))^2 = \sum_{z \in V} (Lg(z))^2 \geq -\lambda_1^{RW}(G) \sum_{z \in V} g(z) Lg(z) = -\lambda_1^{RW} \sum_{z \in V_x} g(z) L_x g(z)$$

We know that $\lambda_1^{RW}(G_x)$ is the largest constant satisfying this inequality for L_x, and we have shown that $\lambda_1^{RW}(G)$ satisfies this inequality as well. We conclude that

$$\lambda_1^{RW}(G_x) \geq \lambda_1^{RW}(G)$$

as desired. □

Lemma 5.4. *For fixed $x \in V$ and $g : V \to \mathbb{R}$, we have*

$$\sum_{y \in V_x} c_{xy} \left[g(y) - g(x)\right]^2 = \sum_{yz \in E_x} c_{y,z}^{*,x} \left[g(y) - g(z)\right]^2 + \frac{1}{\sum_{y \neq x} c_{xy}} \left(Lg(x)\right)^2.$$

Proof. The proof of this theorem involves the Courant-Fisher min-max theorem; see [CLR10, Lemma 2.2] for details. □

Theorem 5.5 (Octopus Inequality)**.** *Define the gradient ∇_{xy} to be the operator*

$$\nabla_{xy} f(\eta) = f(\eta^{xy}) - f(\eta)$$

where $\eta^{xy} = \eta\tau_{xy}$ is the product of η and the transposition of x and y, τ_{xy}. We also define ν as the uniform probability measure on the space of permutations $\mathcal{X}_n$, and $\nu[f] = \int f d\nu$ for functions $f : \mathcal{X}_n \to \mathbb{R}$.

For a weighted graph G on $|V| = n$ vertices, for all $x \in V$ and $f : \mathcal{X} \to \mathbb{R}$, we have

$$\sum_{y \in V_x} c_{xy} \nu\left[(\nabla_{xy} f)^2\right] \geq \sum_{yz \in E_x} c_{yz}^{*,x} \nu\left[(\nabla_{yz} f)^2\right].$$

Proof. The proof of this theorem is beyond the scope of this paper. We refer the reader to [CLR10, Section 3]. □

6. Finishing the Proof of the Conjecture

We now would like to reformulate Aldous's Spectral Gap Conjecture so that it is easier for us to prove. Define

$$\mathcal{H} = \{f : \mathcal{X} \to \mathbb{R} : \nu[f|\xi_i] = 0, i \in V\}$$

where $\nu[f|\xi_i]$ is conditional on the position of the particle labeled i. Let η_x be the label of the particle at x. If $\eta \in \mathcal{X}_n$ satisfies $\xi_i(\eta) = x$, then we have

$$\nu[\cdot|\xi_i](\eta) = \nu[\cdot|\xi_i = x] = \nu[\cdot|\eta_x = i] = \nu[\cdot|\eta_x](\eta)$$

Thus, we can alternatively define $\mathcal{H}$ as

$$\mathcal{H} = \{f : \mathcal{X} \to \mathbb{R} : \nu[f|\eta_x] = 0, x \in V\}$$

Proposition 6.1. *$\mathcal{H}$ contains all eigenvalues in* $\operatorname{Spec}(-\mathcal{L}^{IP}) \setminus \operatorname{Spec}(-\mathcal{L}^{RW})$

Let $\mu^{IP}(G)$ denote the smallest eigenvalue of $-\mathcal{L}^{IP}$ associated to functions only in $\mathcal{H}$. Here, it will be helpful to introduce the Dirichlet form and the related interpretation of the spectral gap.

Definition 6.2. Let P be a reversible transition matrix with stationary distribution π. The **Dirichlet form** associated to P is $\mathcal{E} : \mathbb{R}^{\Omega} \to \mathbb{R}$, given by

$$\mathcal{E}(f, h) = \langle (I - P)f, h\rangle_{\pi} = \sum_{x\in\Omega} (I - P)f(x)h(x)\pi(x)$$

Proposition 6.3.

$$\mathcal{E}(f, f) = \frac{1}{2} \sum_{x,y\in\Omega} (f(x) - f(y))^2 \pi(x)p_{xy}.$$

Proof. We have

$$\begin{aligned}
\mathcal{E}(f, f) &= \langle f, f\rangle_{\pi} - \langle Pf, f\rangle_{\pi} \\
&= \sum_{x\in\Omega} f^2(x)\pi(x) - \sum_{x\in\Omega} Pf(x)f(x)\pi(x) \\
&= \sum_{x\in\Omega} f^2(x)\pi(x) - \sum_{x\in\Omega} f(x)\pi(x) \sum_{y\in\Omega} p_{xy}f(y) \\
&= \sum_{x,y\in\Omega} f^2(x)\pi(x)p_{xy} - f(x)\pi(x)p_{xy}f(y) \\
&= \frac{1}{2} \sum_{x,y\in\Omega} \pi(x)p_{xy}(f(x) - f(y))^2
\end{aligned}$$

as desired. ∎

On our weighted graph, we have

$$\mathcal{E}(f) = \frac{1}{2} \sum_{b\in E} c_b \nu[(\nabla_b(f))^2]$$

and the spectral gap can be described as the largest constant λ such that for all $f : \mathcal{X} \to \mathbb{R}$:

$$\mathcal{E}(f) \geq \lambda \operatorname*{Var}_{\nu}(f) \tag{2}$$

where $\operatorname{Var}_{\nu}(f) = \nu[f^2] - \nu[f]^2$, and $\mu^{IP}(G)$ is the largest such constant if we restrict to $f \in \mathcal{H}$. This shows that

$$\lambda_1^{IP}(G) = \min\{\lambda_1^{RW}(G), \mu_1^{IP}(G)\}$$

We conclude the following:

Corollary 6.4. *Theorem 4.5 is equivalent to the statement that*

$$\mu_1^{IP}(G) \geq \lambda_1^{RW}(G).$$

Proposition 6.5. *For a weighted graph G, we have*

$$\mu_1^{IP}(G) \geq \max_{x\in V} \lambda_1^{IP}(G).$$

Proof. Let $f \in \mathcal{H}$ and $x \in V$. Note that this means that $\nu[f|\mu_x] = 0$, so

$$\nu[f^2] = \operatorname*{Var}_{\nu}(f) = \nu[\operatorname*{Var}_{\nu}(f|\eta_x)].$$

By 2, we know that $\mathcal{E}(f) \geq \lambda_1^{IP}(G)\operatorname{Var}_\nu(f)$, so

$$\lambda_1^{IP}(G_x)\operatorname*{Var}_{\nu}(f|\eta_x) \leq \frac{1}{2}\sum_{b\in E_x}\left(c_b + c_b^{*,x}\nu[(\nabla_b f)^2|\eta_x]\right)$$

We take the expected value of both sides with respect to ν:

$$\lambda_1^{IP}(G_x)\nu[f^2] \leq \frac{1}{2}\sum_{b\in E_x}\left(c_b + c_b^{*,x}\nu[(\nabla_b f)^2]\right)$$

Recall the octopus inequality, which says that

$$\sum_{y\in V_x} c_{xy}\nu\left[(\nabla_{xy} f)^2\right] \geq \sum_{yz\in E_x} c_{yz}^{*,x}\nu\left[(\nabla_{yz} f)^2\right].$$

Applying this to our inequality,

$$\begin{aligned}
\lambda_1^{IP}(G_x)\nu[f^2] &\leq \frac{1}{2}\sum_{b\in E_x}\left(c_b\nu[(\nabla_b f)^2]\right) + \frac{1}{2}\sum_{b\in E_x}\left(c_b^{*,x}\nu[(\nabla_b f)^2]\right)\\
&\leq \frac{1}{2}\sum_{b\in E:x\notin b}\left(c_b\nu[(\nabla_b f)^2]\right) + \frac{1}{2}\sum_{b\in E:x\in b}\left(c_b\nu[(\nabla_b f)^2]\right)\\
&= \frac{1}{2}\sum_{b\in E} c_b\nu[(\nabla_b f)^2] = \mathcal{E}(f).
\end{aligned}$$

We have shown that for all $x \in V$ and $f \in \mathcal{H}$, $\lambda_1^{IP}(G_x)$ satisfies

$$\mathcal{E}(f) \geq \lambda \operatorname*{Var}_{\nu}(f).$$

We know that $\mu_1^{IP}(G)$ is the largest constant that satisfies this inequality, so we can conclude that $\mu_1^{IP}(G) \geq \lambda_1^{IP}(G_x)$ for all $x \in V$ and thus

$$\mu_1^{IP}(G) \geq \max_{x\in V}\lambda_1^{IP}(G_x)$$

as desired. □

We now have all the ingredients we need for the proof of the conjecture.

Proof of Theorem 4.5. We prove the theorem by induction on the number of vertices. Note that when we have two vertices, the random walk and interchange process are actually the same Markov chain, so $\lambda_1^{RW}(G) = \lambda_1^{IP}(G)$.

Now, assume that $\lambda_1^{IP}(G') = \lambda_1^{RW}(G')$ on every weighted graph G' with $n-1$ vertices. Then, if we have some weighted graph G on n vertices, we know that for every $x \in V(G)$, the theorem holds for the reduced network G_x. By Proposition 6.5, we have

$$\mu_1^{IP}(G) \geq \max_{x\in V}\lambda_1^{IP}(G) = \max_{x\in V}\lambda_1^{RW}(G_x)$$

By Proposition 5.2, this is greater than or equal to $\lambda_1^{RW}(G)$. Thus,

$$\mu_1^{IP}(G) \geq \lambda_1^{RW}(G)$$

which by Theorem 5.5 is equivalent to the statement of Aldous's Spectral Gap Conjecture. □

7. Corollaries of Aldous's Spectral Gap Conjecture

Notice how Aldous's Spectral Gap Conjecture allows us to compute eigenvalues of an $n \times n$ matrix (corresponding to the random walk) instead of an $n! \times n!$ matrix (corresponding to the interchange process). We can also deduce some results about other processes in between these two processes.

Corollary 7.1. *In the process of symmetric exclusion on a weighted graph G, we label k of $n = |V|$ particles as occupied and $n-k$ as empty. The k particles are indistinguishable. Our state space is $S^{EP} = \{\zeta \in V : |\zeta| = k\}$, which has $\binom{n}{k}$ elements. Define the process of symmetric exclusion as switching the particles at x and y in our current position ζ if one of x or y is occupied in the state ζ and the other is empty, with probability c^{xy}. The infinitesimal generator of this value is thus*

$$\mathcal{L}^{EP} g(\zeta) = \sum_{xy \in E} c_{xy} (f(\zeta^{xy}) - f(\zeta))$$

where

$$\zeta^{xy} = \begin{cases} \zeta \setminus \{y\} \cup \{x\} & y \in \zeta, x \notin \zeta \\ \zeta \setminus \{x\} \cup \{y\} & x \in \zeta, y \notin \zeta \\ \zeta & \textit{otherwise} \end{cases}$$

Define $\lambda_1^{EP}(G)$ as the largest eigenvalue in $\operatorname{Spec} -\mathcal{L}^{EP}$. *We have*

$$\lambda_1^{EP}(G) = \lambda_1^{RW}(G) = \lambda_1^{IP}(G).$$

Proof. We want to show that

$$\operatorname{Spec}(-\mathcal{L}^{RW}) \subseteq \operatorname{Spec}(-\mathcal{L}^{EP}) \subseteq \operatorname{Spec}(-\mathcal{L}^{IP}).$$

We have $\operatorname{Spec}(-\mathcal{L}^{EP}) \subseteq \operatorname{Spec}(-\mathcal{L}^{IP})$ because the symmetric exclusion process is a subprocess of the interchange process, if we call the particles from 1 to k in the interchange process occupied.

To show that $\operatorname{Spec}(-\mathcal{L}^{RW}) \subseteq \operatorname{Spec}(-\mathcal{L}^{EP})$, we show that if $f : V \to \mathbb{R}$ is an eigenfunction of $-\mathcal{L}^{RW}$ with eigenvalue λ, then λ is an eigenvalue of $-\mathcal{L}^{EP}$ as well. Define $g : S^{EP} \to \mathbb{R}$ as $g(\zeta) = \sum_{x \in \zeta} f(x)$. Then,

$$-\mathcal{L}^{EP} g(\zeta) = -\sum_{x \in \zeta, y \notin \zeta} c_{xy} \left(g\left(\zeta^{xy}\right) - g(\zeta)\right) = -\sum_{x \in \zeta, y \notin \zeta} c_{xy} \left(\sum_{z \in \zeta \setminus \{x\} \cup \{y\}} f(z) - \sum_{z \in \zeta} f(z) \right)$$

Notice that the terms for elements that are not x or y get cancelled out, and that the first sum includes $f(y)$ but not $f(x)$. Thus, this sum is equal to

$$-\sum_{x \in \zeta, y \notin \zeta} c_{xy} \left(f(y) - f(x)\right) \tag{3}$$

Notice that by symmetry,

$$\sum_{x, y \in \zeta, x \neq y} c_{xy} \left(f(y) - f(x)\right) = 0$$

Thus, we can add the same sum over $x \in \zeta, y \in \zeta$ to 3, which yields

$$-\sum_{x \in \zeta, y \neq x} c_{xy} \left(f(y) - f(x)\right) = \sum_{x \in \zeta} -\mathcal{L}^{RW} f(x) = \lambda \sum_{x \in \zeta} f(x) = \lambda g(\zeta)$$

which shows that

$$\mathrm{Spec}(-\mathcal{L}^{RW}) \subseteq \mathrm{Spec}(-\mathcal{L}^{EP}) \subseteq \mathrm{Spec}(-\mathcal{L}^{IP})$$

as desired. We conclude that

$$\lambda_1^{EP}(G) = \lambda_1^{RW}(G) = \lambda_1^{IP}(G).$$

□

We can use the same reasoning for the following similar process:

Corollary 7.2. *In the process of color exclusion, we fix* $n_1, n_2, \ldots, n_r$ *such that* $n_1 + n_2 + \cdots + n_r = n$ *and we assign* $r \geq 2$ *colors* $c_1, c_2, \ldots, c_r$ *to* n *particles such that there are* n_i *particles of color* c_i*. Particles of the same color are indistinguishable. Our state space is the set of partitions* $\alpha = (\alpha_1, \ldots, \alpha_r)$ *where* $|\alpha_i| = n_i$*. In this process, we switch the particles at positions* x *and* y *with probability* c_{xy}*. Our infinitesimal generator for this process is*

$$\mathcal{L}^{CEP} f(\alpha) = \sum_{xy \in E} c_{xy}(f(\alpha^{xy}) - f(\alpha))$$

where we have

$$\alpha^{xy} = \begin{cases} \alpha & x, y \in \alpha_i \text{ for some } i \\ (\alpha_1^{xy}, \ldots, \alpha_r^{xy}) & x \in \alpha_i, y \in \alpha_j, i \neq j \end{cases}, \alpha_k^{xy} = \begin{cases} \alpha_i \setminus \{x\} \cup \{y\} & k = i \\ \alpha_j \setminus \{y\} \cup \{x\} & k = j \\ \alpha_k & k \neq i, k \neq j \end{cases}.$$

Then, we have

$$\mathrm{Spec}(-\mathcal{L}^{RW}) \subseteq \mathrm{Spec}(-\mathcal{L}^{EP} \subseteq \mathrm{Spec}(-\mathcal{L}^{CEP}) \subseteq \mathrm{Spec}(-\mathcal{L}^{IP})$$

and thus

$$\lambda_1^{CEP}(G) = \lambda_1^{RW}(G) = \lambda_1^{IP}(G) = \lambda_1^{EP}(G).$$

We can also use Aldous's Spectral Gap Conjecture to prove bounds on processes involving cycles and matching, where the spectral gap is strictly greater than the spectral gap of the random walk; see [CLR10, Section 6] for a discussion of this.

References

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