MARKOV CHAINS: BROWNIAN MOTION

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ABSTRACT. In this paper we will talk about Brownian Motion and its relationship to Markov Chains.

1. Introduction

Brownian motion, is the random motion of particles suspended in a medium.It is described as random fluctuations in a particle's partition which change the position of the particle and further effects future fluctuations in the closed system. We will talk about stochastic calculus which makes it possible to use calculus on stochastic processes like Brownian motion like the Ito integral Then we will talk about some of the physical interpretations and usage of stochastic and Brownian motion in physics.

2. Definitions

2.1. **Probability Space.** In probability theory, a probability space or a probability triple $(\omega, \alpha.\beta)$ is a mathematical construct that provides a formal model of a random process or "experiment". For example, one can define a probability space which models the throwing of a die.

A probability space consists of three elements:

1). A sample space, Ω, which is the set of all possible outcomes.

2). An event space, which is a set of events α an event being a set of outcomes in the sample space.

3). A probability function, which assigns each event in the event space a probability, which is a number between 0 and 1.

2.2. **Brownian Motion.** Brownian motion, is the random motion of particles suspended in a medium.

This pattern of motion typically consists of random fluctuations in a particle's position inside a fluid sub-domain, followed by a relocation to another sub-domain. Each relocation is followed by more fluctuations within the new closed volume. This pattern describes a fluid at thermal equilibrium, defined by a given temperature. Within such a fluid, there the direction of flow for each particle is random.

2.3. **Stochastic Differential Equations.** A stochastic differential equation is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process. SDEs are used to model various phenomena such as unstable stock prices or physical systems subject to thermal fluctuations. Typically, SDEs

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contain a variable which represents random white noise calculated as the derivative of Brownian motion or the Wiener process.

2.4. **Wiener Process.** In mathematics, the Wiener process is a real valued continuous-time stochastic process named in honor of American mathematician Norbert Wiener for his investigations on the mathematical properties of the one-dimensional Brownian motion. It is one of the best known Lévy processes (stochastic processes with stationary independent increments)

2.5. **Martingale.** In probability theory, a martingale is a sequence of random variables (i.e., a stochastic process) for which, at a particular time, the conditional expectation of the next value in the sequence, regardless of all prior values, is equal to the present value.

2.6. **Semimartingale.** In probability theory, a real valued stochastic process X is called a semimartingale if it can be decomposed as the sum of a local martingale and an adapted finite-variation process. Semimartingales are "good integrators", forming the largest class of processes with respect to which the Itô integral and the Stratonovich integral can be defined.

The class of semimartingales is quite large (including, for example, all continuously differentiable processes, Brownian motion and Poisson processes).

3. Stochastic Calculus

It allows a consistent theory of integration to be defined for integrals of stochastic processes with respect to stochastic processes. It is used to model systems that behave randomly.

The best-known stochastic process to which stochastic calculus is applied is the Wiener process (named in honor of Norbert Wiener), which is used for modeling Brownian motion and other physical diffusion processes in space of particles subject to random forces.

One of the main things used in stochastic calculus is the Ito Integral..

3.1. **Ito Integral.** Itô calculus, named after Kiyoshi Itô, extends the methods of calculus to stochastic processes such as Brownian motion (see Wiener process).

The central concept is the Itô stochastic integral, a stochastic generalization of the Riemann–Stieltjes integral in analysis. The integrands and the integrators are now stochastic processes:

$$Y_t = \int_0^t H_s \, dX_s,$$

3.1.1. *Integration with respect to Brownian Motion.* The Itô integral can be defined in a manner similar to the Riemann–Stieltjes integral, that is as a limit in probability of Riemann sums; such a limit does not necessarily exist pathwise. Suppose that B is a Wiener process (Brownian motion) and that H is a right-continuous , adapted and locally bounded process. If $\{\pi_n\}$ is a sequence of partitions of [0, t] with mesh going to zero, then the Itô integral of H with respect to B up to time t is a random variable

$$\int_0^t H \, dB = \lim_{n\to\infty} \sum_{[t_{i-1},t_i]\in\pi_n} H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}).. \tag{3.1}$$

It can be shown that this limit converges in probability.

For some applications, such as martingale representation theorems and local times, the integral is needed for processes that are not continuous. The predictable processes form the smallest class that is closed under taking limits of sequences and contains all adapted left-continuous processes. If H is any predictable process such that $\int_0^{H2} 0tds < infinity$ for every $t0$ then the integral of H with respect to B can be defined, and H is said to be B-integrable. Any such process can be approximated by a sequence Hn of left-continuous, adapted and locally bounded processes, in the sense that

$$\int_0^t (H - H_n)^2 \, ds \to 0$$

in probability. Then, the Itô integral is

$$\int_0^t H \, dB = \lim_{n \to \infty} \int_0^t H_n \, dB$$

where, again, the limit can be shown to converge in probability. The stochastic integral satisfies the Itô isometry

$$\mathbb{E}\left[\left(\int_0^t H_s \, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 \, ds\right] \tag{3.2}$$

which holds when H is bounded or, more generally, when the integral on the right hand side is finite.

4. Examples

4.1. **Narrow Escape Problem.** The narrow escape problem] is a ubiquitous problem in biology, biophysics and cellular biology.

The mathematical formulation is the following: a Brownian particle (ion, molecule, or protein) is confined to a bounded domain (a compartment or a cell) by a reflecting boundary, except for a small window through which it can escape. The narrow escape problem is that of calculating the mean escape time. This time diverges as the window shrinks, thus rendering the calculation a singular perturbation problem.

When escape is even more stringent due to severe geometrical restrictions at the place of escape, the narrow escape problem becomes the dire strait problem.

4.2. **Ito Diffusion.**

$$\frac{\partial \phi(\mathbf{r}, t)}{\partial t} = \nabla \cdot \big[D(\phi, \mathbf{r}) \ \nabla \phi(\mathbf{r}, t) \big],$$

4.3. **Diffusion Equation.** The diffusion equation is a parabolic partial differential equation. It describes the macroscopic behavior of many micro-particles in Brownian motion. The equation desribes the random movements by the particles after collisions.

4.4. **Continuity Equation.** In physicas is an equation that describes the transport of some quantity. It is particularly simple and powerful when applied to a conserved quantity, but it can be generalized to apply to any extensive quantity. Since mass, energy, momentum, electric charge and other natural quantities are conserved under their respective appropriate conditions, a variety of physical phenomena may be described using continuity equations.

Continuity equations are a generalization of conservation laws.For example, the law of conservation of energy states that energy can neither be created or destroyed. This statement does not rule out the possibility that a quantity of energy could disappear from one point while simultaneously appearing at another point. A stronger statement is that energy is locally conserved: energy can neither be created nor destroyed, nor can it "teleport" from one place to another—it can only move by a continuous flow. A continuity equation is the mathematical way to express this kind of statement. For example, the continuity equation for electric charge states that the amount of electric charge in any volume of space can only change by the amount of electric current flowing into or out of that volume through its boundaries.

Any continuity equation can be expressed in an integral form, which applies to any finite region, or in a "differential form" (in terms of the divergence operator) which applies at a point.

Some examples of continuity equations in physics are the convection–diffusion equation, Boltzmann transport equation, and Navier–Stokes equations. Continuity Equations are one of many examples where Markov Chains and Brownian Motion can be applied.

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