SUMMARY OF RANDOM WALKS ON NETWORKS

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1. Introduction

An application of Markov Chains is to model an electrical network. Through a change in Markov Chains definitions and notations, a movement on the electrical network can be shown through the property of reversibility in a Markov Chain. In this paper, we will be discussing the conductance of an electrical network and the resistance. These can be modeled through a weighted graph, and the Markov Chain would give the probability of moving from one node to another through the wires.
In physics, the movements of electricity can be given by the connections between the wires and battery. The battery has both a positive and negative charge which creates a current to transfer energy through the loop. Here, we will model the source and sink where the source represents the output of energy and the sink essentially represents where the flow of energy is going depending on the charge. The voltage of an electrical network is physically the pressure given by the movement of energy or current which is also the rate of flow or speed of the energy movement. This is a result of the difference between charges. The effective resistance is the total amount of resistance between two points in an electric circuit or network.
We will also discuss common laws such as the Kirchoff's node law, Ohm's Law, Cycle Law, Parallel Law, Series Law, Thompson's principle, and bounds on the effective resistance.

2. Key Definitions and Notations

Before we can explore the implications of Markov chains on electrical networks, we must defines some basic concept to build the connection between physical movements on electrical networks and Markov chains.

Definition 2.1. A network is a finite undirected connected graph G with vertex set V and edge set E, endowed additionally with non-negative numbers known as conductances as defined below.

Definition 2.2. Each edge of the network is assigned a non-negative weight $\{c(e)\}$, called conductances, that are associated to the edges of G. This is analogous with the idea of

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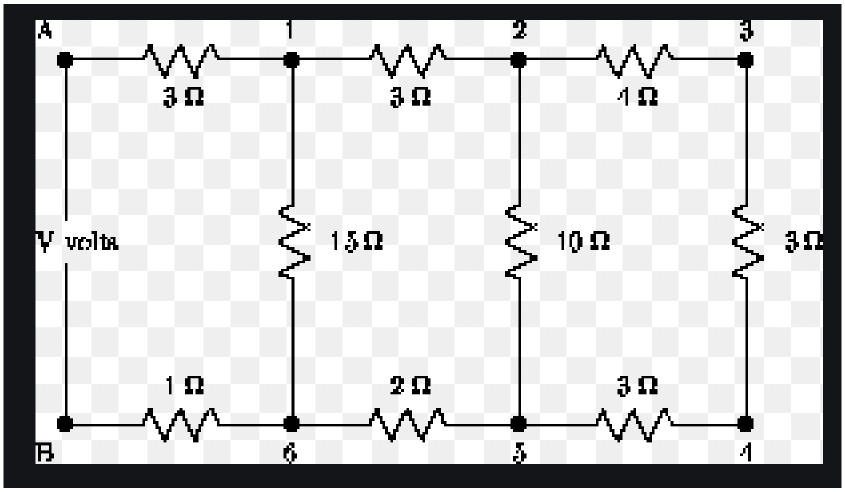

Figure 1. An example of a basic electrical network.

weights assigned on a weighted graph. In the physical definition, this is the ability for a current to flow through an electric network.

Definition 2.3. The reciprocal $r(e) = \frac{1}{c(e)}$ is called the resistance of the edge e. This is the opposition to a current's ability to flow through a network in a system. The resistance is about what deters an electrical current from passing through it. In the figure, the symbol Ω denotes the measurement of this resistance.

Notation. A network will be denoted by the pair $(G, c(e))$. The vertices, G, may also be defined as nodes which can be seen in the dots in the electrical network above.

Let $x, y \in V$. $x \sim y$ means that the vertices x and y are connected by an edge in the network. Consider the Markov chain on the nodes of G with transition matrix $P(x, y) = \frac{c(x,y)}{c(x)}$, where $c(x) = \sum_{y:y\sim x} c(x, y)$ This process is called the weighted random walk on G with edge weights $c(e)$. What happens is that the $P(x, y)$ is the possibility that the current moves to vertex y from x out of all the possible vertices the current can move to from x.

An important property of this Markov Chain is that it is reversible with a probability π defined by $\pi(x) = \frac{c(x)}{c_G}$, where $c_G = \sum_{x\in V} c(x)$.

Proof. $\pi(x)P(x, y) = \frac{c(x)}{c_G}\frac{c(x,y)}{c(x)} = \frac{c(x,y)}{c_G} = \frac{c(y,x)}{c_G} = \frac{c(y)}{c_G}\frac{c(y,x)}{c(y)} = \pi(y)P(y, x)$. ■

We were able to switch $c(x, y)$ to $c(y, x)$ since both denote the same edge between x and y.

Definition 2.4. A function $h : \Omega \to R$ harmonic for a transition matrix P if $h(x) = \sum_{y\in\Omega} P(x, y)h(y)$. In terms of the graph of electrical networks, $h(x)$ is the average of the values at h at neighboring vertices.

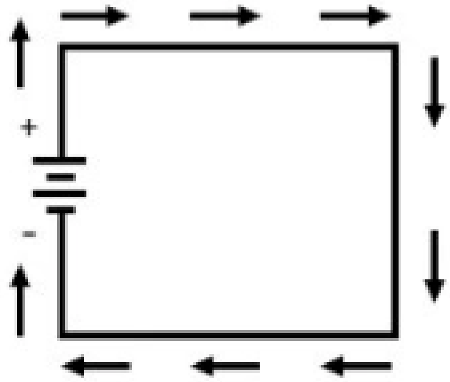

Figure 2. Here is an example of a directed flow on a network. The arrows show the direction of the energy, and the difference between the plus and minus signs cause the movement.

3. Voltage and Flows

Definition 3.1. An oriented edge $e = \vec{xy}$ is an ordered pair of nodes (x, y).

Definition 3.2. A function W which is harmonic on $V \not\{a, z\}$ will be called a voltage. a is the sink of the graph and z is the source. This means that z outputs electricity and a takes in the electricity.

Definition 3.3. A flow θ is a function on oriented edges which is antisymmetric, meaning that $\theta(\vec{xy}) = -\theta(\vec{yx})$.

For a flow θ, define the divergence of θ at x by $div\theta(x) = \sum_{y:y\sim x} \theta(\vec{xy})$. The strength of the flow is $|\theta|$.

Kirchhoff's node law: $\text{div}(x) = 0$ at all $x \notin a, z$, and $\text{div}\,\theta(a) \geq 0$.

This means that the net amount of outputting edges in a vertex that is not the source or sink is 0. Also, the source outputs more oriented edges than what is inputted.

Definition 3.4. Given a voltage W on the network, the current flow I associated with W is defines on oriented edges by $I(\vec{xy}) = \frac{W(x)-W(y)}{r(x,y)} = c(x, y)(W(x) - W(y))$. This is the difference between the voltage at each point scaled by the weight between the vertices. This is actually supported by the physics definition of Ohm's law where the current, I, is proportional to the difference between voltage of two nodes, $W(x) - W(y)$, and inversely proportional to the resistance between those nodes.

Definition 3.5. Cycle Law: If the oriented edges $e_1, \ldots, e_m$ form an oriented cycle (for some $x_0, ..., x_{n-1} \in V$ we have $e_i = (x_{i-1}, x_i)$, where $x_n = x_0$), then $\sum_{i=1}^{m} r(e_i)I(e_i) = 0$. This can be easily explained from the fact that the current between two nodes is anti-symmetric meaning that the sum of flows in between just any two nodes is zero which follows from the Kirchoff's node law.

Proposition 3.6. *(Node law/cycle law/strength). If θ is a flow from a to z satisfying the cycle law, then $\sum_{i=1}^{m} r(e_i)\theta(e_i) = 0$ for any cycle $e_1, ..., e_m$ and if $|\theta| = |I|$, then $\theta = I$.*

Proof. The function $f = \theta \neg I$ satisfies the node law at all nodes and the cycle law. Suppose $f(\rightarrow e_1) > 0$ for some oriented edge $\rightarrow e_1$. By the node law, e_1 must lead to some oriented edge $\rightarrow e_2$ with $f(\rightarrow e_2) > 0$. Iterate this process to obtain a sequence of oriented edges on which f is strictly positive. Since the underlying network is finite, this sequence must eventually revisit a node. The resulting cycle violates the cycle law. ■

4. Effective Resistance

Definition 4.1. Define the effective resistance between vertices a and z by $R(a \leftrightarrow z) = \frac{W(a)-W(z)}{|I|}$.

Definition 4.2. We also define the effective conductance $C(a \leftrightarrow z) = \frac{1}{R(a \leftrightarrow z)}$.

Definition 4.3. The escape probability is $P_a\{T_z < T_a^+\}$. T_z is the first time that the network hits node z. The escape probability refers to the idea that if a current starts at the source, a, then there is a chance that the current may hit the sink before returning to the source.

A result of the escape probability is that for any $x, a, z \in \Omega$,
$P_a T_z < T_a^+ = \frac{1}{c(a)R(a \leftrightarrow z)} = \frac{C(a \leftrightarrow z)}{c(a)}$. [Levin and Wilmer(2009)]

Definition 4.4. The Green's function for a random walk stopped at a stopping time T is defined by $G_T(a, x) = E_\alpha$ (number of visits to x before T) $= E_a(\sum_{t=0}^{\infty} 1_{\{X_t = x, T > t\}}$.

A consequence of the result of the escape probability is that $G_{T_x}(a, x) = c(a)R(a \leftrightarrow z)$. This

Definition 4.5. Parallel Law: Conductances in parallel add: suppose edges e_1 and e_2, with conductances c_1 and c_2, respectively, share vertices v_1 and v_2 as endpoints. Then both edges can be replaced with a single edge of conductance $c_1 + c_2$ without affecting the rest of the network.

Definition 4.6. Series Law: Resistances in series add: if $v \in V \not\{a, z\}$ is a node of degree 2 with neighbors v_1 and v_2, the edges (v_1, v) and (v, v_2) can be replaced by a single edge (v_1, v_2) of resistance $rv_1v + rvv_2$.

Definitions 4.5 and 4.6 give more insight on how the conductances and series act in relation to multiple edges.

Definition 4.7. Define the energy of a flow θ by $E(\theta) := \sum_e [\theta(e)]^2 r(e)$.

Theorem 4.8. *Thompson's Principle: For any finite connected graph, $R(a \leftrightarrow z) = inf\{E(\theta) : \theta$ a unit flow from a to $z\}$. The unique minimizer in the inf above is the unit current flow.*

Proof. Since the set of unit-strength flows can be viewed as a closed and bounded subset of $R|E|$, by compactness there exists a flow θ minimizing $E(\theta)$ subject to $|\theta| = 1$. The compactness just proves the existence of a minimum in the set. Since the map output or flow is also compact, there is a minimum in the set and not an infimum outside the set of values. By Proposition 3.6, to prove that the unit current flow is the unique minimizer, it is enough to verify that any unit flow θ of minimal energy satisfies the cycle law.

Let the edges $e_1, \dots, e_n$ form a cycle. Set $\gamma(e_i) = 1$ for all $1 \leq i \leq n and set \gamma$ equal to zero on all other edges. Note that γ satisfies the node law, so it is a flow, but $\sum \gamma(e_i) = n \neq 0$. For any $\epsilon \in R$, we have by energy minimality that

$0 \leq E(\theta + \epsilon\gamma) \neg E(\theta) = 2\epsilon \sum_{i=1}^{n} r(e_i)\theta(e_i) + O(\epsilon^2)$.

Dividing both sides by $\epsilon > 0$ shows that $0 \leq 2 \sum_{i=1}^{n} r(e_i)\theta(e_i) + O(\epsilon)$,

and letting $\epsilon \downarrow 0$ shows that $0 \leq \sum_{i=1}^{n} r(e_i)\theta(e_i)$. Similarly, dividing by $\epsilon < 0$ and then letting $\epsilon \uparrow 0$ shows that $0 \geq \sum_{i=1}^{n} r(e_i)\theta(e_i)$. Therefore, $0 = \sum_{i=1}^{n} r(e_i)\theta(e_i)$, verifying that θ satisfies the cycle law.

We complete the proof by showing that the unit current flow I has $E(I) = R(a \leftrightarrow z)$ then

$(re)I(e)^2 = \frac{1}{2} \sum_x \sum_y r(x,y)[\frac{W(x)-W(y)}{r(x,y)}]^2$
$= \frac{1}{2} \sum_x \sum_y c(x,y)[W(x) - W(y)]^2$
$= \frac{1}{2} \sum_x \sum_y [W(x) - W(y)]I(xy)$

Since I is antisymmetric,
$\frac{1}{2} \sum_x \sum_y [W(x) - W(y)]I(xy) = \sum_x W(x) \sum_y I(xy)$.
By the node law, $\sum_y I(xy) = 0$ for any $x \notin \{a, z\}$, while $\sum_y I(ay) = |I| = \neg \sum_y I(zy)$, so the right-hand side equals $|I|(W(A) \neg W(z))$.
We know that $|I| = 1$, so the right side is $R(a \leftrightarrow z)$. [Levin and Wilmer(2009)] ■

This shows the connection between the energy of the flow and the effective resistance. One aspect we have not covered is how changing the edges or vertices will affect effective resistance. This will follow from theorems folllowing the Rayleigh's Monotonocity Law.

Theorem 4.9. *If $\{r(e)\}$ and $\{r'(e)\}$ are sets of resistances on the edges of the same graph G and if $r(e) \leq r'(e)$ for all e, then $R(a \leftrightarrow z; r) \leq R(a \leftrightarrow z; r')$.*

Proof. Note that $inf_\theta \sum_e r(e)\theta(e)^2 \leq inf_\theta \sum_e r'(e)\theta(e)^2$ and apply Thomson's Principle. ■

Theorem 4.10. *Adding an edge does not increase the effective resistance $R(a \leftrightarrow z)$. If the added edge is not incident to a, the addition does not decrease the escape probability $P_a T_z < T_a^+ = [c(a)R(a \leftrightarrow z)]^{-1}$.*

Proof. Before we add an edge to a network, we can think of it as existing already with $c = 0$ or $r = \infty$. By adding the edge, we reduce its resistance to a finite number. Combining this with the relationship in definition 4.3 shows that the addition of an edge not incident to a (which we regard as changing a conductance from 0 to a non-zero value) cannot decrease the escape probability $P_a\{T_z < T_a^+\}$. ■

Finally, from the William's Nash inequality, we may find a lower bound on effective resistance. We denote $\prod \subseteq V$ an edge-cutset separating a from z if every path from a to z includes some edge in $\prod$.

Theorem 4.11. *If* $\{\prod_k\}$ *are disjoint edge-cutsets which separate nodes a and z, then* $R(a \leftrightarrow a) \geq \sum_k (\sum_{e\in\prod_k} c(e))^{-1}$.

Proof. Let θ be a unit flow from a to z. For any k, by the Cauchy-Schwarz inequality $\sum_{e\in\prod_k} c(e) \sum_{e\in\prod_k} r(e)\theta(e)^2 \geq (\sum_{e\in\prod_k} \sqrt{c(e)}\sqrt{r(e)}|\theta(e)|)^2$.

The right side is bounded below by $|\theta| = 1$ by definition, so we find that by dividing my the resistance,
$\sum_{e\in\prod_k} r(e)\theta(e)^2 \geq \sum_k \sum_{e\in\prod_k} (r(e))(|\theta(e)|)^2 \geq \sum_k (\sum_{e\in\prod_k} (c(e))^{-1}$. We are done due to the Thompson's Principle. ■

5. Applications and Conclusion

Although the use of random walks on network is a great way of modeling the behavior in an electric circuit, random walks on networks has many other applications.
For example, we may use the series law to prove that the simple walk on integers is recurrent. This is because the effective resistance between the beginning and end is the difference between those integers if we had a set from $\{0, \ldots, m\}$ with reflection at 0 and m. The result of the definition 4.3 shows that as the limit of m increases to infinity, the escape probability approaches 0 making the walk recurrent. Additionally, the Nash-William's inequality may be used to prove how the walk in two dimensions is recurrent. [Lalley(2018)]
To move on, we may also use metrized graphs instead of weighted graphs as in this paper. In metrized graphs,the edges are seen as line segments, and the Laplacian operator on the metrized graphs have interesting properties. Unfortunately, there is background information that I cannot get to such as measure theory. [Baker and Faber(2006)]

References

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