Proof of Pólya's Recurrence Theorem Using Electric Networks

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1 Introduction

A significant result frequently used in probability is Pólya's Recurrence Theorem, which goes as follows:

Theorem 1. *A random walk is recurrent in 1 and 2-dimensional lattices and it is transient for lattices with more than 2 dimensions.*

The initial problem that sparked this theorem, occurred to Pólya in the early 1900's when he crossed the same couple multiple times while taking a walk in the park. Both seemed to be taking random walks.

In this paper, we look at a proof of this theorem that incorporates random walks on electric networks. More specifically, we will investigate the proof provided by Peter Doyle that involves the use of Rayleigh's short-cut method.

Before jumping into the details, we can start by understanding some of the basic intuition. When studying a random walk on a 1-dimensional lattice, one can observe that the probability of returning to the starting point at the second step is simply $\frac{1}{2}$. However, when looking at 2 and 3 dimensional lattices, the probability decreases significantly to $\frac{1}{4}$ and $\frac{1}{6}$. Hence, as the number of points increase, it naturally becomes more challenging for a walker to return to their starting point since there are more paths available. A less intuitive aspect of the recurrence theorem is why there is such a drastic change at $d = 2$. To this day, a more intuitive answer for why random walks become transient for dimensions greater than 2 is yet to be found, however a general explanation can be found in Doyle's [1] work.

2 Preliminary Details

Before proceeding to the main details, it's important that we define certain preliminary terms especially those regarding the physics aspect of this paper.However, please keep in mind that the terms we define today, will be defined mostly within the mathematical context that they will be used for. For more physical definitions, please refer to other sources.

An electric network, is simply a finite connected graph with two disjoint sets of vertices: vertices grounded and connected to the negative pole of a battery and vertices connected to the positive pole. Some basic physics intuition can provide with the fact that having such connected vertices allows for the flow of electricity. Networks also have the following few important properties as defined below:

- **Voltage:** Voltage is a harmonic function $v : V(G) \to \mathbb{R}^+$ that assigns a positive number to each vertex x in a network G.
- **Current:** Current is the function $i : E(G)^+ \to \mathbb{R}$ that assigns a number to each "oriented edge" $(x, y) \in E(G)^+$. Usually denoted as i_{xy} , current also follows the following property: $i_{xy} = -i_{yx}$, which is why we have the current pay attention to the orientation of edges.
- **Resistance:**Resistance is a function $E(G) \to \mathbb{R}^+$ that assigns a positive number (in ohms, Ω) to each unoriented edge $\{x, y\} \in E(G)$, denoted as R_{xy}
- **Conductance** For now we will define conductance as the reciprocal of resistance i.e. $C_{xy} = 1/R_{xy},$. Additionally, we will define the conductance of a vertex x to be the sum of the conductances of the edges leaving it: i.e. $C_x = \sum_{y \in N(x)} C_{xy}$.
- **Effective Resistance** The effective Resistance of a two points in a circuit, is the ratio between the total voltage and the total current flowing between the two points. We will denote this as R_{eff}

3 Implementing Random Walks on Networks

In order to prove Pólya's Theorem [theorem 1], we need to find a relationship between escape probabilities and effective resistance through the use of random walks. To symbolise a simple random walk on a graph, we can simply use an electric circuit with a 1V battery applied between the starting point, a, and end point b. Additionally we place a 1Ω resistor at each edge. (Applying a battery creates a flow of electricity, and resistors simply resists flow, meaning it slows down or terminates flow in sections of the circuit).

When implementing random walks on a circuit, it is fair for us to use the voltage as the probability of a walker reaching a point 1V before a point 0V. Now, we can express the escape probability (the probability the walker starting at point a reaches point b) as:

$$p_{\text{escape}} = 1 - p_{\text{return}} = 1 - \sum_x P_{ax} p_x$$

where p_{return} is the probability that the walker returns to point a. This is equivalent to the sum of the probability that the walker reaches point a before point b from a point x, provided that the walker reached x from point a.

As we defined earlier, the effective resistance between two points is the ration between the total voltage and total current. In calculating the amount of current flowing between point a and b, we can notice that there is only current flow from point a to b so we get $I_{ab} = I_a$. Then, when we calculate this value, we get:

$$\begin{aligned}
I_a &= \sum_x (V_a - V_x)\, C_{ax} \\
&= V_a \sum_x C_{ax} - \frac{C_a}{C_a} \sum_x C_{ax} V_x \\
&= C_a - C_a \sum_x \frac{C_{ax}}{C_a} V_x \\
&= C_a \left(1 - \sum_x P_{ax} V_x\right) \implies I_a = C_a p_{\text{escape}}
\end{aligned}$$

Using this result, combined with the fact that $V_x = p_x$ we have:

$$R_{eff} = \frac{V_{ab}}{I_{ab}} = \frac{V_{ab}}{C_a p_{\text{escape}}} \implies p_{\text{escape}} = \frac{V_{ab}}{C_a R_{eff}}$$

Hence, we now have the relationship we were looking for between escape probabilities and effective resistance.

4 Proof of Pólya's Theorem

To begin our proof, we want to start by taking an infinite lattice and converting it into a finite graph $G^{(r)}$ with radius r, by doing the following: Mark the starting point of the random walk as the origin and then proceed to get rid of all edges that are more than r edges away from the origin. Taking our graph, $G^{(r)}$ we can denote the points that are r edges away from the origin (the points at the extremities) as $S^{(r)}$. Now, as we increase r to infinity, we turn our finite graph into an infinite one. A random walk on such graph $G^{(r)}$ begins at the origin and ends when a point in S^r is reached (refer to figure 1 for an example). We can then denote the escape probability of $G^{(r)}$ as $p^{(r)}_{\text{escape}}$ and the escape probability of the infinite graph as $\lim_{r\to\infty} p^{(r)}_{\text{escape}}$.

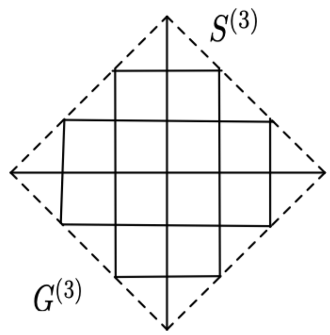

Figure 1: example of $G^{(3)}$

Now, if we proceed to turn $G^{(r)}$ into an electric circuit as we described earlier, we will attach our 1V battery such that the origin will have a voltage of 1V, and the points in $S^{(r)}$ have a voltage of 0V. Using our previously derived equation, we can then see that:

$$p^{(r)}_{\text{escape}} = \frac{1}{C_a R^{(r)}_{eff}} = \frac{1}{2dR^{(r)}_{eff}}$$

where C_a is the total conductance at the origin. Now if we were to take the limit of this equation to find the escape probability for the infinite graph (as well as the effective resistance of an infinite circuit) we get the following:

$$p_{\text{escape}} = \frac{1}{2dR_{eff}}.$$

Now, looking at what we get for the escape probability, it is clear that in order to have $p_{\text{escape}} = 0$ our effective resistance must be infinity. Hence, it suffices to show that if the effective resistance of a lattice is infinity, then the random walk is recurrent.

4.1 Recurrence in 1 and 2 Dimensions

First we will show that the effective resistance for a 1D circuit is ∞.

An infinite 1D circuit is simply formed by a chain of resisters for which it can be seen that the effective resistance to infinity is in fact infinity. This can be seen, because of the rotational symmetry of the lattice. However, since 2D lattices are not rotationally symmetric, it is not as simple to deduce the effective resistance for them.

For a 2D lattice, we must use Rayleigh's law of monotonicity, which in essence is equivalent to the shorting and cutting law. The laws go as follows:

- **Shorting Law** Shorting certain sets of nodes together can only decrease the effective resistance of the network between two given nodes.
- **Cutting Law** Cutting certain branches can only increase the effective resistance between two given nodes.
- **Monotonicity Law** The effective resistance between two given nodes is monotonic in the branch resistances.

If we short the edges of a 2-D lattice, we can easily compute the resistance to infinity as follows. Resistors between two levels are in parallel and would give one resistor of value $\frac{1}{8n+4}\Omega$ where n is the value of the lower level (refer to figure 2). Hence, the resistance to infinity is $\sum_{n=0}^{\infty} \frac{1}{8n+4} = \infty$. Since shorting, as stated in the law, on decreases the effective resistance, the actual resistance to infinity of the 2D lattice is ∞.

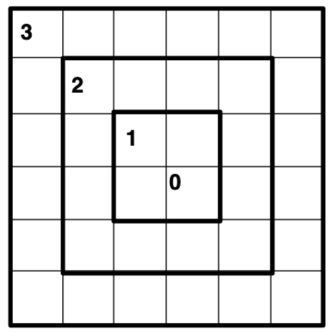

Figure 2: levels in 2D lattice

4.2 Transience in 3 dimension

Now, we want to show that $p_{\text{escape}} > 0 \implies R_{eff} < \infty$ in order to prove that a walk on a 3D lattice is transient. We will use a similar approach as we did for the 2D lattice, however, since we want to calculate an upper bound, we will use the cutting theorem instead of the shorting theorem.

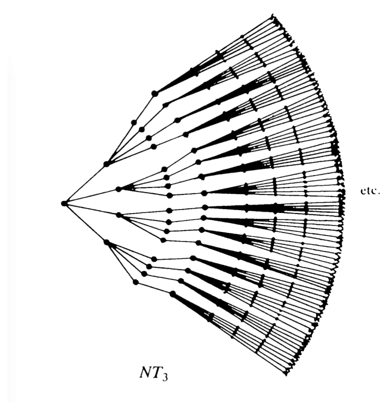

Figure 3: NT_3

In order to construct a model we can work with, we will use a tree that grows as slowly as the 3D lattice. Instead of dividing the tree into branches at the edge, we will allow the tree to branch each time the radius doubles in size and since in 3D lattice size of sphere quadruples, we make four branches.

We call this tree NT_3 (refer to figure 3). Now, when computing R_{eff} for NT_3, we use the fact that by symmetry, the voltage at each level in the tree is the same. So if we cut these points we can calculate the resistance to infinity of NT_3 as $\frac{1}{2}$.

Now, when we try to embed NT_3 back into a 3D lattice, the tree that we actually end up embedding, is $NT_{2.5849}$, meaning it is of the 2.5849 dimension (details on embedding trees are available in [2] Doyle and Snell). We say this is because the number of nodes in a ball of radius r for this tree is proportional to $r^{2.5849}$. The R_{eff} for $NT_{2.5849}$ is 1. What this means, is that we have a graph from a 3D resistance to infinity, which in fact is finite, and is equal or larger than that of a 3D lattice. Hence, it is in fact fair to say, that the resistance to infinity of an infinite 3D lattice, is finite, and therefor the random walk on the 3D lattice must be transient.

Remark. Another related proof that one could study, is the one provided by [3] Prasad Tetali. Tetali also looks at the recurrence problem as as an electric circuit problem problem. However, in his proof, instead of applying a 1V voltage to the circuit as Doyle did, Tetali instead passes a 1 amp current through the circuit. The only difference that doing this will provide to the proof, is how to form the connection between effective resistance and random walks.

Remark. It's only natural to be curious, as to how the use random walks can connect back to Pólya's question regarding see the couple walking in the park. If one were to use Pólya's theorem, they could show that two walkers taking a random walk on a 2D lattice are certain to meet, by combing the two random walks into one. Then, this combined random walk takes two steps each time, and it suffices to show that the random walk crosses the origin.

References

[1] Peter G Doyle. *Application of Rayleigh's short-cut method to Polya's recurrence problem.* PhD thesis, Dartmouth, 1982.

[2] Peter G. Doyle and J. Laurie Snell. *Random Walks and Electric Networks.* Mathematical Association of America, 1984.

[3] Prasad Tetali. Random walks and the effective resistance of networks. *Journal of Theoretical Probability*, 4(1):101–109, 1991.