JORDAN CANONICAL FORM

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Primary Decomposition

Theorem 0.1. *(Primary decomposition) Let $T \in L(\mathbb{C}^n)$ with distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then there is a direct sum decomposition*

$$V = K_{\lambda_1} + K_{\lambda_2} + \cdots + K_{\lambda_n}$$

with respect to which T has a block matrix

$$\begin{pmatrix} _1 \cdot I + N_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 \cdot I + N_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \cdot I + N_n \end{pmatrix}$$

where each N_j is nilpotent.

Proof: We will prove three lemmas first.

(a)

Lemma 0.2. *Let N be a nilpotent r by r matrix, M an s by s matrix and F an arbitrary r by s matrix. Then the matrix equation $X - NXM = F$.*

Proof: Let $k+1$ be the degree of the matrix. When $X = F + NFM + N^2FM^2 + ... + N^kFM^k$, the equation is solved.

(b)

Lemma 0.3. *If λ is an eigenvalue of the s by s matrix A and N is a nilpotent r by r matrix,*

$$A = \begin{pmatrix} \lambda \cdot I + N & E \\ 0 & A \end{pmatrix}$$

and

$$B = \begin{pmatrix} \lambda \cdot I + N & 0 \\ 0 & A \end{pmatrix}$$

are similar matrices.
Proof: Let

$$P = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$$

It is sufficient to show that there is a matrix X such that $AP = PB$, which is true when $X(A - \lambda \cdot I) - NX = E$ or $X - NX(A - \lambda \cdot I)^{-1} = E(A - \lambda \cdot I)^{-1}$,

Date: October 2020.

which is solvable according to Lemma 0.2.

(c)

Lemma 0.4. *Every operator $T \in L(\mathbb{C}^n)$ with eigenvalue λ has a matrix of the form*

$$A = \begin{pmatrix} \lambda \cdot I + N & 0 \\ 0 & A \end{pmatrix}$$

Proof: We use induction on n, the dimension of the space. case n=1 is trivial. Assume the statement holds for case n-1, we show that it's true for case n. Let $\{e_1, e_2, \cdots, e_{n-1}\}$ be the basis of a n-1 dimension vector space, then we have a matrix representing

$$M_{n-1} = \begin{pmatrix} \lambda \cdot I + N & 0 \\ 0 & A \end{pmatrix}$$

. We now add another vector e_n to our basis such that $\{e_1, e_2, \cdots, e_{n-1}, e_n\}$ forms a basis of the n-dimensional vector space, and let $\lambda_1 = T(e_1)$. If λ_1 is not an eigenvalue of M_{n-1}, then the new matrix

$$M_n = \begin{pmatrix} \lambda_1 \cdot I + N & E \\ 0 & M_{n-1} \end{pmatrix}$$

By lemma 0.3, this is similar to the matrix

$$M_n = \begin{pmatrix} \lambda_1 \cdot I + N & 0 \\ 0 & M_{n-1} \end{pmatrix}$$

If λ_1 is an eigenvalue of M_{n-1}, then by the induction hypothesis, we have

$$M_n = \begin{pmatrix} \lambda_1 & E_1 & E_2 \\ 0 & \lambda_1 \cdot I & 0 \\ 0 & 0 & A \end{pmatrix}$$

which is similar to

$$M_n = \begin{pmatrix} \lambda_1 & E_1 & 0 \\ 0 & \lambda_1 \cdot I & 0 \\ 0 & 0 & A \end{pmatrix}$$

by lemma 0.3. To prove the Primary decomposition, simply apply lemma 0.4 inductively.

Jordan Decomposition

Definition 0.5. Let V be a linear space over complex numbers. A chain of generalized eigenvectors for a linear function $f : V \to V$ with eigenvalue λ is a sequence of non-zero vectors $v_1, \cdots, v_k$ such that $f(v_1) = \lambda \cdot v_1$ and $f(v_i) = \lambda \cdot v_i + v_{i-1}$ for $i = 2, 3, \cdots, k$

Lemma 0.6. *Let V be a linear space over complex numbers of finite dimension n. For every function $f : V \to V$, there exists a chain of generalized eigenvectors $C_1, C_2, \cdots, C_m$ such that the union of $C_1, C_2, \cdots, C_m$ is a basis of V.*

Definition 0.7. a Jordan Canonical Form is an upper triangular matrix consisting of J_λ, called Jordan blocks, where each J_λ takes the form

$$J_\lambda = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \cdots & \lambda \end{pmatrix}$$

, and the whole Jordan matrix takes the form

$$J = \begin{pmatrix} J_{\lambda 1} & 0 & 0 & \cdots & 0 \\ 0 & J_{\lambda 2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & J_{\lambda k} \end{pmatrix}$$

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Theorem 0.8. *Every square matrix A can is similar to a matrix in Jordan Normal Form.*

Proof: Let $Ax = f(x)$ and $C_1, C_2, \cdots, C_m$ be the chains of generalized eigenvectors of f, forming a basis called B of $\mathbb{C}^n$. Let $C_1 = v_1, v_2, \cdots, v_k$, then $f(v_i) = \lambda \cdot v_i + v_{i-1}$. Thus the k-th column of $[f]_{B,B}$ is $\lambda \cdot e_i + e_{i-1}$, thus a Jordan matrix.

Reference

https://iuuk.mff.cuni.cz/ rakdver/linalg/lesson15-8.pdf

https://www.csie.ntu.edu.tw/ b89089/link/*jordan.pdf*