Lattice Percolation on $\mathbb{Z}^d$

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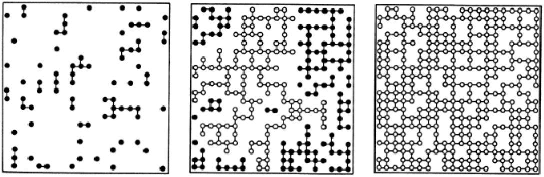
Phase Transition

Porous Medium $\longrightarrow$ Random Graph on $\mathbb{Z}^d$



$p = 0.2$ $p = 0.59$ $p = 0.8$

The Fundamental Question

Let each edge be open with i.i.d. probability p, and let $\theta(p) = P_p(|C_0| = \infty)$ be the probability of an infinite open component. Does there exist a $p_c \in (0, 1)$ such that:

$$\theta(p) = 0 \text{ for } p < p_c \quad \text{and} \quad \theta(p) > 0 \text{ for } p > p_c?$$

The Mathematical Model

Definition 1 (The Lattice $\mathbb{L}^d$)

The graph is $\mathbb{L}^d = \langle \mathbb{Z}^d, \mathbb{E}^d \rangle$, where the set of vertices is the integer lattice $\mathbb{Z}^d$. The set of edges $\mathbb{E}^d$ consists of all unordered pairs $\{x, y\}$ of vertices in $\mathbb{Z}^d$ with l_1-distance $\delta(x, y) = \sum_{i=1}^{d} |x_i - y_i| = 1$.

Definition 2 (The Probability Space)

The sample space is the set of all bond configurations $\Omega = \{0, 1\}^{\mathbb{E}^d}$. For a given $p \in [0, 1]$, the product Bernoulli measure P_p is defined on Ω such that for any edge e, the probability of its state being 1 (open) is $P_p(\omega(e) = 1) = p$, independently of all other edges.

Kolmogorov's 0-1 Law

Tail Events

Let $\{\mathcal{F}_n\}$ be a sequence of σ-algebras generated by a sequence of independent random variables. The tail σ-algebra is $\mathcal{T} = \bigcap_{n=1}^{\infty} \sigma(\bigcup_{k=n}^{\infty} \mathcal{F}_k)$. An event $A \in \mathcal{T}$ is a **tail event**.

Theorem 3 (Kolmogorov's 0-1 Law)

If A is a tail event, then $P(A) \in \{0, 1\}$.

Application: The event $S = \{\omega \in \Omega \mid \exists \text{ an infinite open cluster}\}$ is a tail event with respect to the edge states. Therefore, $P_p(S) \in \{0, 1\}$.

The Critical Probability p_c

The percolation probability is defined as $\theta(p) = P_p(|C_0| = \infty)$, where C_0 is the open cluster containing the origin.

Definition 4 (Critical Probability)

$p_c(d)$ is the threshold where $\theta(p)$ becomes non-zero.

$$p_c(d) = \sup\{p \mid \theta(p) = 0\} = \inf\{p \mid \theta(p) > 0\}$$

The relationship $\theta(p) > 0 \iff P_p(S) = 1$ ensures the transition is sharp.

Property I: $p_c(1) = 1$

Theorem 5 ($p_c(1) = 1$)

For bond percolation on $\mathbb{Z}^1$, the critical probability is 1.

Proof.

For $p < 1$, let A_n be the event that edge $\langle n, n+1 \rangle$ is closed. The events $\{A_n\}$ are independent with $P_p(A_n) = 1 - p > 0$. Since

$$\sum_{n=1}^{\infty} P_p(A_n) = \sum_{n=1}^{\infty} (1 - p) = \infty$$

the second Borel-Cantelli Lemma implies $P_p(A_n \text{ i.o.}) = 1$. Almost surely, infinitely many edges to the right (and left) of the origin are closed, preventing an infinite path from the origin. Thus, $\theta(p) = 0$ for all $p < 1$. Since $\theta(1) = 1$, we conclude that $p_c(1) = 1$. $\square$

Property II: Monotonicity

Theorem 6 (Monotonicity)

The critical probability is non-increasing in dimension: $p_c(d+1) \leq p_c(d)$ *for* $d \geq 1$*.*

Proof.

Let $\theta_d(p)$ be the percolation probability on $\mathbb{L}^d$. We embed $\mathbb{L}^d$ in $\mathbb{L}^{d+1}$ via the map $(x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_d, 0)$. A configuration ω on $\mathbb{L}^{d+1}$ induces a process on the embedded $\mathbb{L}^d$. An infinite open cluster in $\mathbb{L}^d$ is also an infinite open cluster in $\mathbb{L}^{d+1}$. This gives the event inclusion:

$$\{|C_0(\omega)| = \infty \text{ on } \mathbb{L}^d\} \subseteq \{|C_0(\omega)| = \infty \text{ on } \mathbb{L}^{d+1}\}$$

This immediately implies $\theta_d(p) \leq \theta_{d+1}(p)$ for all p. From the definition $p_c(d) = \inf\{p \mid \theta_d(p) > 0\}$, the result follows. □

Main Theorem: Non-Trivial Transition for $d \geq 2$

Theorem 7 (Non-Degeneracy)

For any dimension $d \geq 2$, the critical probability is non-degenerate:

$$0 < p_c(d) < 1$$

This guarantees the existence of distinct subcritical and supercritical phases in higher dimensions.

Proof Strategy: $p_c(d) > 0$ via Path Counting

Rationale

The event $\{|C_0| = \infty\}$ implies the existence of an infinite open *self-avoiding* path from the origin. By counting these simpler structures, we can find an upper bound on $\theta(p)$.

Let N_n be the number of open self-avoiding paths of length n from 0.

$$\theta(p) \leq P_p(N_n \geq 1) \leq E_p[N_n] = \sigma_n(d) \cdot p^n$$
$$\leq (2d(2d-1)^{n-1}) \cdot p^n = \frac{2d}{2d-1} \cdot [p(2d-1)]^n$$

Conclusion: If $p(2d-1) < 1$, the RHS $\to 0$ as $n \to \infty$, forcing $\theta(p) = 0$. Thus $p_c(d) \geq \frac{1}{2d-1} > 0$.

The Duality Principle in $\mathbb{L}^2$

For the planar lattice $\mathbb{L}^2$, we define a dual lattice $\mathbb{L}^{2,*}$ whose vertices are at the centers of the faces of $\mathbb{L}^2$.

Definition 8 (Dual Configuration)

Each edge $e \in \mathbb{E}^2$ is crossed by exactly one dual edge e^*. We define the state of the dual configuration ω^* by:

$$\omega^*(e^*) = 1 - \omega(e)$$

A dual edge is open iff the primal edge is closed. Percolation on the dual lattice occurs with parameter $1 - p$.

Key Topological Fact

An open cluster in the primal lattice $\mathbb{L}^2$ is finite if and only if it is enclosed by an open path (a circuit) in the dual lattice $\mathbb{L}^{2,*}$.

Proof Strategy: $p_c(2) < 1$ using Duality

Goal: Show $\exists p < 1$ such that $\theta(p) > 0$.

Using the duality principle:

$$\begin{aligned} 1 - \theta(p) &= P_p(|C_0| < \infty) \\ &= P_p(\text{origin is enclosed by an open dual circuit } \gamma^*) \\ &\leq \sum_{\text{circuits } \gamma^*} P_p(\gamma^* \text{ is open}) \\ &= \sum_{n=4}^{\infty} \rho_n (1-p)^n \end{aligned}$$

Conclusion: The power series in $(1-p)$ has a positive radius of convergence. For p sufficiently close to 1, the sum is less than 1, which implies $1 - \theta(p) < 1$ and thus $\theta(p) > 0$.

Equivalence of Critical Points

Definition 9 (Mean Cluster Size)

A key thermodynamic quantity is the mean cluster size, defined as $\chi(p) = E_p[|C_0|]$. This gives rise to another critical point, $p_T = \sup\{p \mid \chi(p) < \infty\}$.

Theorem 10 (Uniqueness of the Critical Point)

For any dimension d, the geometric and thermodynamic critical points coincide:

$$p_c(d) = p_T(d)$$

Proof Strategy: $p_c = p_T$ via Exponential Decay

Goal: Show $p < p_c \implies \chi(p) < \infty$.

The key result for $p < p_c$ is the exponential decay of connectivity:

$$P_p(\text{origin connects to distance } n) \leq e^{-nh(p)} \quad \text{for some } h(p) > 0.$$

This allows bounding the mean cluster size by summing over cluster radii:

$$\chi(p) = \sum_{k=1}^{\infty} k \cdot P_p(|C_0| = k) \leq \sum_{n=0}^{\infty} |S(n)| \cdot P_p(\text{radius is } n)$$
$$\leq \sum_{n=0}^{\infty} (C \cdot n^d) \cdot e^{-nh(p)} < \infty$$

The convergence follows because exponential decay overcomes polynomial growth.

Advanced Tools: The Role of Correlation Inequalities

- **FKG Inequality**: $P_p(A \cap B) \geq P_p(A)P_p(B)$ for increasing events A, B. It is fundamental for proving monotonicity (e.g., $\theta(p)$ is non-decreasing) and for showing that connectivity-enhancing events are positively correlated. Intuitively speaking, it formalizes that "more open edges help".
- **BK Inequality**: $P_p(A \circ B) \leq P_p(A)P_p(B)$. It provides upper bounds on probabilities of events requiring disjoint resources.
- **Russo's Formula**: $\frac{d}{dp}P_p(A) = E_p[N_{\text{pivotal}}(A)]$.

Uniqueness of the Infinite Cluster

Theorem 11

For $p > p_c(d)$ with $d \geq 2$, there is almost surely ***exactly one*** *infinite open cluster.*

Proof Outline

Let N be the number of infinite open clusters (IOCs). By the 0-1 Law, $P_p(N = k) = 1$ for some constant $k \in \{0, 1, \dots, \infty\}$. Since $p > p_c$, we know $k \geq 1$. The proof shows $k = 1$ by demonstrating that both $k \in \{2, 3, \dots\}$ and $k = \infty$ are impossible.

Proof of Uniqueness, Part I: Ruling out Finite Plurality

Goal: Show $P_p(N \geq 2) = 0$, which rules out $k \in \{2, 3, \dots\}$.

Lemma 12

Let M_B be the number of distinct IOCs intersecting a finite set B. Then $P_p(M_B \geq 2) = 0$.

Proof.

Let $\{B_n\}_{n=1}^{\infty}$ be an increasing sequence of finite sets with $\cup_n B_n = \mathbb{Z}^d$. Then $\{N \geq 2\} = \cup_{n=1}^{\infty}\{M_{B_n} \geq 2\}$. By the continuity of probability measure:

$$P_p(N \geq 2) = P_p(\lim_{n\to\infty}\{M_{B_n} \geq 2\}) = \lim_{n\to\infty} P_p(M_{B_n} \geq 2) = 0$$

□

Proof of Uniqueness, Part II: The Contradiction Setup

Goal: Show $P_p(N = \infty) = 0$ by contradiction.

The Argument's Core Idea

If there were infinitely many infinite clusters, the lattice would be "dense" with points where these clusters meet. We can show this density is geometrically impossible.

Step 1: The Assumption and its Consequence

Assume there are infinitely many infinite clusters ($P_p(N = \infty) = 1$).
A direct consequence is that special vertices called **trifurcation points** must exist with positive probability. A trifurcation point is a vertex where at least three distinct infinite clusters are accessible.
This leads to a necessary condition:

$$P_p(\text{the origin is a trifurcation point}) > 0$$

Proof of Uniqueness, Part III: The Geometric Contradiction

Goal: Show that the consequence $P_p(\text{origin is a trifurcation}) > 0$ is impossible.

Step 2: The Geometric Constraint

Consider a large box $B(n)$ of side length $2n + 1$.
The expected number of trifurcation points inside the box is proportional to its volume:

$$E[\# \text{ of trifurcations in } B(n)] = |B(n)| \cdot P_p(\text{origin is a trifurcation}) \propto n^d$$

However, a combinatorial argument shows that the number of such points is also limited by the box's surface area:

$$E[\# \text{ of trifurcations in } B(n)] \leq C \cdot |\partial B(n)| \propto n^{d-1}$$

Step 3: The Contradiction

Combining these gives the inequality:

$$|B(n)| \cdot P_p(\text{origin is a trifurcation}) \leq C \cdot n^{d-1}$$

$$\implies P_p(\text{origin is a trifurcation}) \leq \frac{C \cdot n^{d-1}}{|B(n)|} \approx \frac{C \cdot n^{d-1}}{C' \cdot n^d} = O\left(\frac{1}{n}\right)$$

As we let the box grow ($n \to \infty$), the right side goes to 0. This forces the probability to be 0, which contradicts our necessary condition.

Thanks for listening!