An Explicit Evaluation of the Rogers-Ramanujan Continued Fraction

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Euler Circle

July 12, 2025

Infinite Continued Fractions

(Convergence)

$$b_0 + \cfrac{a_0}{b_1 + \cfrac{a_1}{b_2 + \cfrac{a_2}{b_3 + \ddots}}}$$

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$$b_0 = b_0, \quad b_0 + \frac{a_0}{b_1} = \frac{b_0 b_1 + a_0}{b_1}, \quad b_0 + \cfrac{a_0}{b_1 + \cfrac{a_1}{b_2}} = \frac{b_0 b_1 b_2 + a_1 b_0 + a_0 b_2}{b_1 b_2 + a_1}, \ldots$$

Examples

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \ddots}}}},$$

$$\frac{1}{e-1} = \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \ddots}}}}}.$$

Golden Ratio

(Simplest infinite continued fraction)

$$\frac{1}{\phi} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}},$$

where

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

Rogers-Ramanujan Continued Fraction

Definition

$$R(q) = \cfrac{q^{\frac{1}{5}}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \ddots}}}}, \qquad |q| < 1.$$

Rogers-Ramanujan Continued Fraction

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Ramanujan:

$$R(e^{-2\pi}) = \cfrac{e^{-\frac{2\pi}{5}}}{1 + \cfrac{e^{-2\pi}}{1 + \cfrac{e^{-4\pi}}{1 + \ddots}}} = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}.$$

Fast Convergence

$$\frac{e^{-\frac{2\pi}{5}}}{1+\frac{e^{-2\pi}}{1+e^{-4\pi}}} = 0.284,079,043,840,412,308,\ldots$$

$$\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2} = 0.284,079,043,840,412,296,\ldots$$

Infinite Series, Products, and Continued Fractions

Euler:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 \pi^2}\right) \implies \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

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Rogers and Ramanujan:

$$\cfrac{1}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \ddots}}} = \frac{1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(1-q)\dots(1-q^n)}}{1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)\dots(1-q^n)}}$$

$$= \prod_{n=1}^{\infty} \frac{(1 - q^{5n-1})(1 - q^{5n-4})}{(1 - q^{5n-2})(1 - q^{5n-3})}$$

Rogers-Ramanujan Identities

Theorem (Rogers-Ramanujan Identities)

$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{(1-q)\dots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-2})(1-q^{5n-3})},$$
$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)\dots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})}.$$

Theta Functions

(Jacobi Triple Product Identity and Theta Function Transformations)

Definition (Theta Function)

For every $z \in \mathbb{C}$ and $\Im(\tau) > 0$,

$$\vartheta_3(z, \tau) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz}$$

where $q = e^{\pi i \tau}$.

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Theorem (**Jacobi Triple Product Identity**)

$$\vartheta_3(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2iz})(1+q^{2n-1}e^{-2iz}).$$

Theorem (Theta Function Transformation)

$$\vartheta_3(0,\tau) = \frac{1}{\sqrt{-i\tau}}\vartheta_3(0,-1/\tau),$$

i.e.

$$\sum_{n=-\infty}^{\infty} e^{n^2\pi i\tau} = (-i\tau)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-\frac{n^2\pi i}{\tau}}.$$

Important Formula for $R(q)$

Using Jacobi Triple Product Identity,

Theorem

$$\frac{1}{R(q)} - 1 - R(q) = q^{-\frac{1}{5}} \prod_{n=1}^{\infty} \left(\frac{1 - q^{\frac{n}{5}}}{1 - q^{5n}} \right).$$

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Using theta function transformation,

Lemma

$$\frac{1}{R(e^{-2\pi})} - 1 - R(e^{-2\pi}) = \sqrt{5}.$$

Evaluating $R(e^{-2\pi})$

$$\frac{1}{x} - 1 - x = \sqrt{5}$$

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$$x^2 + 2\phi x + \phi^2 = \phi\sqrt{5}$$

$$x = \sqrt{\phi\sqrt{5}} - \phi$$

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}.$$

More Results

Theorem

$$-R(-e^{-\pi}) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}.$$

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Theorem (Modular Equation of Degree 5)

If $v = R(q)$ and $u = R(q^5)$, then

$$v^5 = u\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}.$$

Even More Results

Theorem

$$R(e^{-2\pi\sqrt{5}}) = \frac{\sqrt{5}}{1+\left(5^{\frac{3}{4}}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{5}{2}}-1\right)^{\frac{1}{5}}} - \frac{\sqrt{5}+1}{2}.$$

Theorem

If $\alpha\beta = \pi^2$, *then*

$$\left\{\frac{\sqrt{5}+1}{2} + R(e^{-2\alpha})\right\}\left\{\frac{\sqrt{5}+1}{2} + R(e^{-2\beta})\right\} = \frac{5+\sqrt{5}}{2}.$$

Further Reading

Bruce C Berndt and George E Andrews.
Ramanujan's Lost Notebook.
Part I. New York: Springer, 2005.

Bruce C. Berndt.
Number theory in the spirit of Ramanujan, volume 34 of *Stud. Math. Libr.*
Providence, RI: American Mathematical Society (AMS), 2006.

Edmund Taylor Whittaker and George Neville Watson.
A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions with an account of the principal transcendental functions. Edited and prepared for publication by Victor H. Moll. With a new foreword by S. J. Patterson.
Cambridge: Cambridge University Press, 5th edition edition, 2021.

Thanks

Thank you for your attention! 

Acknowledgements

The author would like to thank Simon Rubinstein-Salzedo and Benjamin Vakil.