AN EXPLICIT EVALUATION OF THE ROGERS-RAMANUJAN CONTINUED FRACTION.

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Abstract. In this paper we provide a proof of Ramanujan's first non-elementary explicit evaluation of the Rogers-Ramanujan Continued Fraction using the Jacobi Triple Product Identity and the theta function transformations as our main ingredients. We also prove the celebrated Rogers-Ramanujan identities. A theme of the subject is that the interchange between any two of infinite series, infinite products and infinite continued fractions often leads to beautiful results in mathematics.

1. Introduction.

The story of how Ramanujan amazed Hardy with theorems on continued fractions, among which was included the explicit evaluation

$$\frac{e^{-\frac{2\pi}{5}}}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{1+\ddots}}}} = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}, \tag{1.1}$$

is very famous.

In this paper we will provide a proof of (1.1), using the Jacobi Triple Product Identity 4.1 and the theta function transformations 6.1 as our main ingredients. A moderate understanding of complex analysis, in particular Liouville's Theorem and analytic continuation, and a basic understanding of Fourier series are assumed. Moreover, the reader is expected to be quite comfortable when dealing with infinite products as well as infinite series.

Let x equal the right side of (1.1). Basic algebraic manipulations show that

$$\frac{1}{x} - 1 - x = \sqrt{5}. \tag{1.2}$$

The last equation is a special case of the more general relationship:

$$\frac{1}{R(q)} - 1 - R(q) = q^{-\frac{1}{5}} \prod_{n=1}^{\infty} \left(\frac{1-q^{\frac{n}{5}}}{1-q^{5n}} \right), \tag{1.3}$$

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*This is an anagram of the author's real name.

proved in Section 7, where $R(q)$ denotes the Rogers-Ramanujan continued fraction:

$$R(q) := \cfrac{q^{\frac{1}{5}}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\ddots}}}, \tag{1.4}$$

where $q = e^{\pi i \tau}$ for any $\Im(\tau) > 0$.
By putting $q = e^{-2\pi}$ in (1.3), and comparing with (1.2), we see that we are required to prove that

$$e^{\frac{2\pi}{5}} \prod_{n=1}^{\infty} \left(\frac{1 - e^{-\frac{2n\pi}{5}}}{1 - e^{-10n\pi}} \right) = \sqrt{5}. \tag{1.5}$$

In fact, (1.5) is obtained by putting $\tau = \frac{i}{5}$ in the more general transformation

$$e^{\frac{\pi i}{2}\left(\tau + \frac{1}{\tau}\right)} \prod_{n=1}^{\infty} \left(\frac{1 - e^{2n\pi i \tau}}{1 - e^{-\frac{2n\pi i}{\tau}}} \right)^6 = \frac{1}{i\tau^3} \tag{1.6}$$

obtained in Section 6 using the theta function transformations.
We will now indicate a prominent theme of this paper. Conversions between any two of infinite series, infinite products and infinite continued fractions often lead to beautiful results in mathematics. For example, it is well-known how the equation

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 \pi^2} \right)$$

leads to the resolution of the Basel problem

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

In this paper we will repeatedly be converting between infinite series, infinite products, and infinite continued fractions.
After establishing some preliminary notions and definitions in the next section, we discuss the convergence of the Rogers-Ramanujan continued fraction (1.4) and also convert it into the quotient of two infinite series in Section 3 by using the Rogers-Ramanujan functions.
This is followed in Section 4 by a short complex analytical proof of the Jacobi Triple Product Identity, an immensely important result that gives a product representation for the theta functions.
Then in Section 5 we express $R(q)$ as an infinite product by proving the celebrated Rogers-Ramanujan identities.
In Section 6 we establish the important theta function transformations, used to prove (1.5).
Finally, we will be able to prove Ramanujan's evaluation of $R(e^{-2\pi})$ in Section 7.
In the last section we state more theorems about $R(q)$ and provide references to the literature for the proofs.

2. Preliminary notions.

By convention, an empty sum is zero, while an empty product is 1.
Throughout the entire paper, q and τ will be related by $q = e^{\pi i \tau}$ with τ having positive imaginary part. We also define $q' = e^{\pi i \tau'}$ where $\tau'\tau = -1$. These definitions will be used freely from now on.
While $R(q)$, defined by (1.4), converges on $|q| < 1$, we will only prove its convergence for $0 < q < 1$ in Section 3, because this will be sufficient for our purposes. Of primary importance in this paper will be the theta functions, which we now define.

Definition 2.1 (Theta Functions). For every $z \in \mathbb{C}$,

$$\vartheta_4(z,q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz}$$

$$\vartheta_3(z,q) := \vartheta_4\left(z + \frac{\pi}{2}, q\right) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz}$$

$$\vartheta_2(z,q) := e^{\frac{\pi i \tau}{4}} \cdot e^{iz}\vartheta_4\left(z + \frac{\pi}{2} + \frac{\pi\tau}{2}, q\right) = \sum_{n=-\infty}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} e^{(2n+1)iz}$$

Remark 2.2. If we denote $\Im(\tau) = t > 0$ and we suppose that $|z| \leq M$ for arbitrary positive real numbers M, then

$$|\vartheta_3(z,q)| \leq \sum_{n=-\infty}^{\infty} |e^{\pi i \tau n^2 + 2niz}| \leq \sum_{n=-\infty}^{\infty} e^{-\pi t n^2 + 2nM}.$$

Since the last series converges, it follows that $\vartheta_3(0,q)$ is an entire function of z for every fixed $\Im(\tau) > 0$, and is a holomorphic function of q in the unit disk $|q| < 1$ for every fixed $z \in \mathbb{C}$. The same is true, of course, for the other theta functions.

Immediate consequences of the previous definition are stated in the next corollary.

Corollary 2.3.

$$\vartheta_2(z+\pi,q) = -\vartheta_2(z,q) \qquad \vartheta_2(z+\pi\tau,q) = e^{-\pi i\tau}\cdot e^{-2iz}\vartheta_2(z,q)$$
$$\vartheta_3(z+\pi,q) = \vartheta_3(z,q) \qquad \vartheta_3(z+\pi\tau,q) = e^{-\pi i\tau}\cdot e^{-2iz}\vartheta_3(z,q)$$
$$\vartheta_4(z+\pi,q) = \vartheta_4(z,q) \qquad \vartheta_4(z+\pi\tau,q) = -e^{-\pi i\tau}\cdot e^{-2iz}\vartheta_4(z,q)$$

Also following immediately from Definition 2.1 is the corollary which is stated next.

Corollary 2.4.

$$\vartheta_2(z,q) = e^{\frac{\pi i\tau}{4}}\cdot e^{iz}\vartheta_3\left(z+\frac{1}{2}\pi\tau, q\right) = e^{\frac{\pi i\tau}{4}}\cdot e^{iz}\vartheta_4\left(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau, q\right)$$

$$\vartheta_3(z,q) = \vartheta_4\left(z+\frac{1}{2}\pi, q\right) = e^{\frac{\pi i\tau}{4}}\cdot e^{iz}\vartheta_2\left(z+\frac{1}{2}\pi\tau, q\right)$$

$$\vartheta_4(z,q) = ie^{\frac{\pi i\tau}{4}}\cdot e^{iz}\vartheta_2\left(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau, q\right) = \vartheta_3\left(z+\frac{1}{2}\pi, q\right)$$

In this paper we will prove the Rogers-Ramanujan identities. Accordingly, we define the Rogers-Ramanujan functions next.

Definition 2.5 (**Rogers-Ramanujan functions**).

$$G(q) := 1 + \sum_{k=1}^{\infty} \frac{q^{k^2}}{(1-q)\dots(1-q^k)}$$
$$H(q) := 1 + \sum_{k=1}^{\infty} \frac{q^{k(k+1)}}{(1-q)\dots(1-q^k)}$$

3. Convergence of $R(q)$.

It is not the purpose of this paper to develop a whole theory of continued fractions. An excellent book discussing general convergence properties, among other things, of these interesting mathematical objects, is [LW08].
Our discussion of the Rogers-Ramanujan continued fraction $R(q)$ in this paper will be restricted to positive real values of q such that $0 < q < 1$. Accordingly, when we discuss the convergence of $R(q)$, we will not consider complex values of q, although $R(q)$ does actually converge in the open unit disk $|q| < 1$. The reader interested in the latter fact will find a useful reference in [Ber06, Chapter 7].

Theorem 3.1. *The continued fraction*

$$1 + \cfrac{aq}{1 + \cfrac{aq^2}{1 + \ddots}}$$

converges for all positive real values a and q such that $a > 0$ and $0 < q < 1$.

Remark 3.2. Actually the result is valid for all $a \in \mathbb{C}$ and all complex values q in the open unit disk $|q| < 1$; but as we have already mentioned before, we are not considering complex values of a and q in this paper.

We need to prove that the sequence of approximants to the continued fraction is convergent. Accordingly, the proof of Theorem 3.1 relies on the following lemma:

Lemma 3.3. *If, for every positive integer n,*

$$\mu_n := 1 + \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(1-q)\dots(1-q^{n-k+1})a^k q^{k^2}}{\{(1-q)\dots(1-q^k)\}\{(1-q)\dots(1-q^{n-2k+1})\}},$$
$$\nu_n := 1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(1-q)\dots(1-q^{n-k})a^k q^{k(k+1)}}{\{(1-q)\dots(1-q^k)\}\{(1-q)\dots(1-q^{n-2k})\}},$$

then,

$$\frac{\mu_n}{\nu_n} = 1 + \cfrac{aq}{1 + \cfrac{aq^2}{1 + \ddots + \cfrac{aq^n}{1}}}.$$

Proof. We need to define a new series: for every positive integer n and every integer $0 \leq r \leq n$,

$$F_{n,r} := 1 + \sum_{k=1}^{\lfloor \frac{n-r+1}{2} \rfloor} \frac{(1-q)\dots(1-q^{n-r-k+1})a^k q^{k(r+k)}}{\{(1-q)\dots(1-q^k)\}\{(1-q)\dots(1-q^{n-r-2k+1})\}}.$$

Consequently, $F_{n,0} = \mu_n$ and $F_{n,1} = \nu_n$. Also $F_{n,n} = 1$ and $F_{n,n-1} = 1 + aq^n$. Moreover, since $a > 0$ and $0 < q < 1$, it immediately follows that $F_{n,r} \geq 1$, so that division by $F_{n,r} \neq 0$ is permitted. We want to develop $\mu_n/\nu_n = F_0/F_1$ into a continued fraction by finding a recurrence relation for $F_{n,r}$ on r for every fixed n. To achieve this we consider the difference

$$\begin{aligned} F_{n,r} - F_{n,r+1} = & \sum_{k=1}^{\lfloor \frac{n-r+1}{2} \rfloor} \frac{(1-q)\dots(1-q^{n-r-k+1})a^k q^{k(r+k)}}{\{(1-q)\dots(1-q^k)\}\{(1-q)\dots(1-q^{n-r-2k+1})\}} \\ & - \sum_{k=1}^{\lfloor \frac{n-r}{2} \rfloor} \frac{(1-q)\dots(1-q^{n-r-k})a^k q^{k(r+1+k)}}{\{(1-q)\dots(1-q^k)\}\{(1-q)\dots(1-q^{n-r-2k})\}}, \end{aligned}$$

where $0 \leq r \leq n-2$. Now two cases arise according to whether $n-r$ is odd or even. First, suppose that $n-r$ is odd.The right side becomes

$$\begin{aligned} & \sum_{k=1}^{\frac{n-r+1}{2}} \frac{(1-q)\dots(1-q^{n-r-k+1})a^k q^{k(r+k)}}{\{(1-q)\dots(1-q^k)\}\{(1-q)\dots(1-q^{n-r-2k+1})\}} \\ & - \sum_{k=1}^{\frac{n-r-1}{2}} \frac{(1-q)\dots(1-q^{n-r-k})a^k q^{k(r+1+k)}}{\{(1-q)\dots(1-q^k)\}\{(1-q)\dots(1-q^{n-r-2k})\}}, \end{aligned} \tag{3.1}$$

and we can combine the two sums by temporarily separating the ultimate summand of the first sum to obtain

$$\begin{aligned} & \sum_{k=1}^{\frac{n-r-1}{2}} \frac{(1-q)\dots(1-q^{n-r-k})a^k q^{k(r+k)}}{\{(1-q)\dots(1-q^k)\}\{(1-q)\dots(1-q^{n-r-2k})\}} \left(\frac{1-q^{n-r-k+1}}{1-q^{n-r-2k+1}} - q^k \right) \\ & \quad + a^{\frac{n-r+1}{2}} q^{\frac{n+r+1}{2}\left(r+\frac{n-r+1}{2}\right)} \\ & = aq^{r+1} + \sum_{k=2}^{\frac{n-r+1}{2}} \frac{(1-q)\dots(1-q^{n-r-k})a^k q^{k(r+k)}}{\{(1-q)\dots(1-q^{k-1})\}\{(1-q)\dots(1-q^{n-r-2k+1})\}} \\ & = aq^{r+1} + a^k q^{r+1} \sum_{k=1}^{\frac{n-r-1}{2}} \frac{(1-q)\dots(1-q^{n-(r+2)-k+1})a^k q^{k(r+2+k)}}{\{(1-q)\dots(1-q^k)\}\{(1-q)\dots(1-q^{n-(r+2)-2k+1})\}} \\ & = aq^{r+1} F_{n,r+2}. \end{aligned}$$

We reach the same result, for the case where $n-r$ is even, by following much the same procedure except that (3.1) must first be replaced by

$$\sum_{k=1}^{\frac{n-r}{2}} \frac{(1-q)\dots(1-q^{n-r-k})a^k q^{k(r+k)}}{\{(1-q)\dots(1-q^k)\}\{(1-q)\dots(1-q^{n-r-2k})\}} \left(\frac{1-q^{n-r-k+1}}{1-q^{n-r-2k+1}} - q^k \right).$$

It would be futile to supply these calculations inasmuch as the reader will have no difficulty in filling in these straightforward and easy steps.
We have therefore shown that, for $0 \leq r \leq n-2$,

$$F_{n,r} - F_{n,r+1} = aq^{r+1}F_{n,r+2}.$$

Equipped with this recurrence, we now develop μ_n/ν_n into a finite continued fraction as follows:

$$\frac{\mu_n}{\nu_n} = \frac{F_{n,0}}{F_{n,1}} = \frac{F_{n,1} + aqF_{n,2}}{F_{n,1}} = 1 + \frac{aq}{F_{n,1}/F_{n,2}} = 1 + \cfrac{aq}{1 + \cfrac{aq^2}{1 + \ddots + \cfrac{aq^{n-1}}{F_{n,n-1}/F_{n,n}}}}$$

$$= 1 + \cfrac{aq}{1 + \cfrac{aq^2}{1 + \ddots \cfrac{aq^{n-1}}{1 + \cfrac{aq^n}{1}}}}.$$

■

Corollary 3.4. *For any $a > 0$ and $0 < q < 1$,*

$$\frac{1 + \sum_{k=1}^{\infty} \frac{a^k q^{k^2}}{(1-q)\dots(1-q^k)}}{1 + \sum_{k=1}^{\infty} \frac{a^k q^{k(k+1)}}{(1-q)\dots(1-q^k)}} = 1 + \cfrac{aq}{1 + \cfrac{aq^2}{1 + \ddots}}$$

Proof. This is apparent by letting $n \to \infty$ in the previous lemma. We show that the infinite series in the numerator and denominator are convergent. This is easily achieved:

$$\sum_{k=1}^{\infty} \frac{a^k q^{k^2}}{(1-q)\dots(1-q^k)} \leq \sum_{k=1}^{\infty} \frac{a^k q^{k^2}}{(1-q)^k}.$$

Now the series on the right converges by the ratio test. We similarly prove the convergence of $\sum_{k=1}^{\infty} \frac{a^k q^{k(k+1)}}{(1-q)\dots(1-q^k)}$. ■

Theorem 3.1 is now an immediate consequence of Corollary 3.4.

Corollary 3.5. *Recall Definition 2.5. We have*

$$R(q) = q^{\frac{1}{5}} \frac{H(q)}{G(q)}.$$

Proof. This is an immediate consequence of the previous corollary and the definition of $R(q)$ by (1.4). ■

4. The Jacobi Triple Product Identity.

We have already alluded to the Jacobi Triple Product Identity in the introduction. This powerful result is perhaps the most important result in the theory of theta functions and, in fact, gives product representations for the theta functions in Definition 2.1. All known proofs of Theorem 7.2, namely [Hir00, Wat29b], utilise the Jacobi Triple Product Identity.

Theorem 4.1 (Jacobi Triple Product Identity). *For all $z \in \mathbb{C}$,*

$$\vartheta_3(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2iz})(1+q^{2n-1}e^{-2iz}).$$

We will give a short proof of Theorem 4.1 by first proving two lemmas due to Euler [Eul51] after discussing the holomorphic properties of an infinite product.

Lemma 4.2. *Let us denote the infinite product $\prod_{m=0}^{\infty}(1+y^m\omega)$ by $F(\omega)$ when $|y|<1$. Then $F(\omega)$ is an entire function of ω for every fixed $|y|<1$.*

Proof. Let $|\omega| < M$ for an arbitrary positive real number M. Since $|y| < 1$, there exists a minimum non-negative integer M_0 for which $M|y|^m < \frac{1}{2}$ whenever $m > M_0$. For every $m > M_0$, we have, if we make use of the principal branch of the logarithm,

$$|\log(1+\omega y^m)| \leq M|y|^m \sum_{n=1}^{\infty} \frac{M^{n-1}|y|^{m(n-1)}}{n} \leq M|y|^m \sum_{n=1}^{\infty} \frac{1}{n\cdot 2^{n-1}} \leq 2M|y|^m$$

and since $\sum_{m=M_0+1}^{\infty}|y|^m$ converges, it follows that $\sum_{m=M_0+1}^{\infty}\log(1+y^m\omega)$ is holomorphic on $|\omega|<M$ if $|y|<1$ is fixed. From this it follows that

$$\exp\left(\sum_{m=M_0+1}^{\infty}\log(1+y^m\omega)\right) = \prod_{m=M_0+1}^{\infty}(1+y^m\omega),$$

and hence $\prod_{m=0}^{\infty}(1+y^m\omega)$, is an entire function of ω for every fixed $|y|<1$. ■

Lemma 4.3. *For all $\omega \in \mathbb{C}$ and all y such that $|y|<1$, we have*

$$\prod_{m=0}^{\infty}(1+y^m\omega) = \sum_{r=0}^{\infty} \frac{y^{\frac{r(r-1)}{2}}\omega^r}{(1-y)\dots(1-y^r)}.$$

Proof. We denote the Taylor expansion of the left side, which, by the previous lemma, is an entire function of ω provided that $|y|<1$, by

$$F(\omega) = \sum_{n=0}^{\infty} a_n\omega^n. \tag{4.1}$$

Then it immediately follows that $a_0 = 1$. But now

$$F(\omega y) = \prod_{m=0}^{\infty}(1+\omega y^{m+1}),$$

so that

$$(1+\omega)F(\omega y) = F(\omega).$$

Therefore

$$\sum_{n=0}^{\infty} a_n\omega^n y^n + \sum_{n=1}^{\infty} a_{n-1}\omega^n y^{n-1} = \sum_{n=0}^{\infty} a_n\omega^n.$$

Equating coefficients of ω on both sides gives

$$a_n y^n + a_{n-1}y^{n-1} = a_n,$$

so that we have the recurrence relation

$$a_n = \frac{a_{n-1}y^{n-1}}{1-y^n}$$

for $n \geq 1$. Since $a_0 = 1$, it follows that

$$a_n = \frac{y^{0+1+\cdots+(n-1)}}{(1-y)\cdots(1-y^n)} = \frac{y^{n(n-1)/2}}{(1-y)\dots(1-y^n)}.$$

The lemma now follows by replacing this value of a_n in (4.1). ■

Lemma 4.4. *For all $|y| < 1$ and $|\omega| < 1$, we have*

$$\prod_{m=0}^{\infty}(1+y^m\omega)^{-1} = \sum_{r=0}^{\infty} \frac{(-1)^r\omega^r}{(1-y)\dots(1-y^r)}.$$

Proof. Inasmuch as the proof of this lemma is very similar to that of Lemma 4.3, we leave it as an exercise to the reader. ■

We can now prove Theorem 4.1.

Proof. In Lemma 4.3, put $y = q^2$ and $\omega = qe^{2iz}$ to find

$$\begin{aligned}\prod_{n=1}^{\infty}(1+q^{2n-1}e^{2iz}) = \sum_{n=0}^{\infty}\frac{e^{2niz}q^{n^2}}{(1-q^2)\dots(1-q^{2n})} &= \sum_{n=0}^{\infty}\frac{e^{2niz}q^{n^2}(1-q^{2n+2})(1-q^{2n+4})\dots}{(1-q^2)(1-q^4)\dots} \\ &= \left[\prod_{j=0}^{\infty}(1-q^{2j+2})^{-1}\right]\left[\sum_{n=0}^{\infty}e^{2niz}q^{n^2}\prod_{m=0}^{\infty}(1-q^{2n+2+2m})\right] \\ &= \left[\prod_{j=0}^{\infty}(1-q^{2j+2})^{-1}\right]\left[\sum_{n=-\infty}^{\infty}e^{2niz}q^{n^2}\prod_{m=0}^{\infty}(1-q^{2n+2+2m}),\right]\end{aligned}$$

because $\prod_{m=0}^{\infty}(1-q^{2n+2+2m})$ equals zero whenever n is a negative integer. By making use of Lemma 4.3 again but this time by putting $y = q^2$, and $\omega = -q^{2n+2}$, the last expression equals

$$\left[\prod_{j=0}^{\infty}(1-q^{2j+2})^{-1}\right]\left[\sum_{n=-\infty}^{\infty}e^{2niz}q^{n^2}\sum_{r=0}^{\infty}\frac{(-1)^r q^{(2n+2)r+r^2-r}}{(1-q^2)\dots(1-q^{2r})}\right],$$

which can be written as

$$\left[\prod_{j=0}^{\infty}(1-q^{2j+2})^{-1}\right]\left[\sum_{n=-\infty}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r e^{2niz}q^{(n+r)^2+r}}{(1-q^2)\dots(1-q^{2r})}.\right]$$

We can invert the double sum provided that it converges absolutely. For any fixed q such that $|q| = R$ where $R < 1$, we have

$$\begin{aligned}\left|\sum_{r=0}^{\infty}\frac{(-1)^r e^{2inz}q^{(n+r)^2+r}}{(1-q^2)\dots(1-q^{2r})}\right| &\leq \sum_{r=0}^{\infty}\frac{|e^{2inz}|R^{(n+r)^2+r}}{(1-R^2)^r} \\ &\leq |e^{2inz}|R^{n^2}\sum_{r=0}^{\infty}\frac{R^{r^2+r}}{(1-R^2)^r} = C|e^{2inz}|R^{n^2},\end{aligned}$$

for some C independent of n. Since $\sum_{n=-\infty}^{\infty} |e^{2inz}|R^{n^2}$ is convergent, the inversion of the double sum is justified, so that

$$\prod_{n=0}^{\infty}(1+q^{2n+1}e^{2iz}) = \left[\prod_{j=0}^{\infty}(1-q^{2j+2})^{-1}\right]\left[\sum_{r=0}^{\infty}\frac{(-1)^r e^{-2irz}q^r}{(1-q^2)\dots(1-q^{2r})}\sum_{n=-\infty}^{\infty} e^{2i(n+r)z}q^{(n+r)^2}\right].$$

The ultimate sum being absolutely convergent, we can re-index it by changing n into $n-r$ without altering the value and the right side can be factorised as

$$\left[\prod_{j=0}^{\infty}(1-q^{2j+2})^{-1}\right]\left[\sum_{r=0}^{\infty}\frac{(-qe^{-2iz})^r}{(1-q^2)\dots(1-q^{2r})}\right]\left[\sum_{n=-\infty}^{\infty} e^{2inz}q^{n^2}\right].$$

But now if we restrict $|qe^{-2iz}| < 1$, we can make use of Lemma 4.4, with $y = q^2$ and $\omega = qe^{-2iz}$ to deduce that

$$\prod_{n=0}^{\infty}(1+q^{2n+1}e^{2iz}) = \left[\prod_{j=0}^{\infty}(1-q^{2j+2})^{-1}\right]\left[\prod_{k=0}^{\infty}(1+q^{2n+1}e^{-2iz})^{-1}\right]\left[\sum_{n=-\infty}^{\infty} e^{2inz}q^{n^2}\right],$$

i.e.

$$\sum_{n=-\infty}^{\infty} q^{n^2}e^{2inz} = \prod_{n=0}^{\infty}(1+q^{2n+1}e^{2iz})(1+q^{2n+1}e^{-2iz})(1-q^{2n+2}),$$

when $|q| < 1$ and $|q| < |e^{2iz}|$. But both sides of the previous equality are entire functions of z for every fixed $|q| < 1$, by Lemma 4.2 and Remark 2.2. Since, moreover, both sides agree on a set of values of z, namely $|e^{2iz}| > |q|$, a subset of which is the open lower half z-plane, containing at least one limit point, therefore both sides agree on the entire complex z-plane for every fixed $|q| < 1$, by analytic continuation. ■

It is now a straightforward exercise to give infinite product representations for the other theta functions.

Corollary 4.5. *For all $z \in \mathbb{C}$,*

$$\vartheta_4(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1}e^{2iz})(1-q^{2n-1}e^{-2iz}),$$

$$\vartheta_3(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2iz})(1+q^{2n-1}e^{-2iz}),$$

$$\vartheta_2(z,q) = 2e^{\frac{\pi i\tau}{4}}\cos z\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n}e^{-2iz})(1+q^{2n}e^{-2iz}).$$

Proof. These follow from a direct application of Theorem 4.1 in Definition 2.1. ■

Two special cases of the Jacobi Triple Product Identity will be especially useful in this paper.

Corollary 4.6. *For $|x| < 1$,*

$$\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(5n+1)}{2}} = \prod_{n=1}^{\infty}(1-x^{5n-3})(1-x^{5n-2})(1-x^{5n}).$$

Proof. Let $q = e^{5\pi iu}$ and $z = (1+u)\pi/2$, where $\Im(u) > 0$, in Theorem 4.1. This gives

$$\sum_{n=-\infty}^{\infty} e^{5\pi iun^2} e^{\pi in(1+u)} = \prod_{n=1}^{\infty}(1-e^{10\pi iun})(1+e^{5(2n-1)\pi iu+\pi i(1+u)})(1+e^{5(2n-1)\pi iun-\pi i(1+u)}),$$

or, letting $x = e^{2\pi iu}$,

$$\sum_{n=-\infty}^{\infty} e^{\pi in} x^{\frac{5n^2+n}{2}} = \prod_{n=1}^{\infty}(1-x^{5n})(1-x^{5n-2})(1-x^{5n-3}),$$

as desired. ■

Corollary 4.7. *For* $|x| < 1$*,*

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n+3)}{2}} = \prod_{n=1}^{\infty}(1-x^{5n-4})(1-x^{5n-1})(1-x^{5n}).$$

Proof. The proof proceeds similarly as in the last corollary, except that now we put $q = e^{5\pi iu}$ and $z = (1+3u)\pi/2$. ■

Remark 4.8. By absolute convergence, we find that

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n+1)}{2}} = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n-1)}{2}},$$

by changing n into $-n$, when $|x| < 1$. Similarly

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n+3)}{2}} = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n-3)}{2}}.$$

We will use these representations interchangeably in the rest of the paper without further comment.

5. The Rogers-Ramanujan Identities.

The goal of this section is to express $R(q)$ as an infinite product. We will do this in Theorem 5.7 but first we must prove the Rogers-Ramanujan Identities. These give product representations for the Rogers-Ramanujan functions $G(q)$ and $H(q)$ from Definition 2.5.

Theorem 5.1 (Rogers-Ramanujan Identities).

$$G(q) = \prod_{n=1}^{\infty}(1-q^{5n+1})^{-1}(1-q^{5n+4})^{-1}, \qquad H(q) = \prod_{n=1}^{\infty}(1-q^{5n+2})^{-1}(1-q^{5n+3})^{-1}.$$

Remark 5.2. The perceptive reader who may have noticed some similarity between the Rogers-Ramanujan Identities and Lemma 4.4 is encouraged to learn about more this by researching on *Ramanujan pairs*, references to which can be found in [Ber85, p. 130].

We give a sequence of lemmas, in the first of which we define a function $\mathcal{G}(x)$ which generalises the infinite sum of Corollary 4.6 which equals $\mathcal{G}(1)$. We will subsequently find a recurrence, of a similar type as that found in the proof of 4.3, for a closely related function which will then be expressible as a series which generalises both the Rogers-Ramanujan functions $G(q)$ and $H(q)$. This will enable us to prove Theorem 5.1. Corollary 3.4 will then be used to express $R(q)$ as an infinite series.

Lemma 5.3. *If, for every $x \in \mathbb{C}$, we define*

$$\mathcal{G}(x) := 1 + \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{\frac{n(5n-1)}{2}} (1 - xq^{2n}) \frac{(1-xq)(1-xq^2)\dots(1-xq^{n-1})}{(1-q)(1-q^2)\dots(1-q^n)},$$

then

$$\mathcal{G}(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2}n(5n+1)} (1 - x^2 q^{4n+2}) \frac{(1-xq)\dots(1-xq^n)}{(1-q)(1-q^2)\dots(1-q^n)}.$$

Remark 5.4. Although we have used a similar notation, the functions $\mathcal{G}$ and G are obviously different.

Proof.

$$\mathcal{G}(x) = 1 + \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2}n(5n-1)} \frac{(1-xq)(1-xq^2)\dots(1-xq^{n-1})}{(1-q)(1-q^2)\dots(1-q^{n-1})}$$
$$+ \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2}n(5n+1)} \frac{(1-xq)(1-xq^2)\dots(1-xq^n)}{(1-q)(1-q^2)\dots(1-q^n)}.$$

By taking away the first summand of the first infinite series on the right and re-indexing, we deduce that

$$\mathcal{G}(x) = 1 - x^2q^2 + \sum_{n=1}^{\infty} (-1)^n \left(x^{2n} q^{\frac{n(5n+1)}{2}} - x^{2n+2} q^{\frac{(n+1)(5n+4)}{2}} \right) \frac{(1-xq)\dots(1-xq^n)}{(1-q)\dots(1-q^n)}$$
$$= 1 - x^2q^2 + \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2}n(5n+1)} (1 - x^2 q^{4n+2}) \frac{(1-xq)\dots(1-xq^n)}{(1-q)(1-q^2)\dots(1-q^n)}.$$

■

Lemma 5.5. *For $\mathcal{G}$ defined as in the previous lemma,*

$$\frac{\mathcal{G}(x)}{1-xq} - \mathcal{G}(xq) = xq(1 - xq^2)\mathcal{G}(xq^2).$$

Proof. If we denote the right side by $\mathcal{H}(x)$, then, using the previous lemma

$$\mathcal{H}(x) = xq$$
$$+ \frac{1}{1-xq} \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n (1-xq)\dots(1-xq^n)}{(1-q)(1-q^2)\dots(1-q^n)} x^{2n} q^{\frac{1}{2}n(5n+1)} \left[(1 - x^2q^{4n+2}) - q^n(1 - xq^{2n+1}) \right] \right\}$$
$$= xq + \frac{1}{1-xq} \sum_{n=1}^{\infty} (-1)^n \frac{(1-xq)\dots(1-xq^n)}{(1-q)\dots(1-q^{n-1})} x^{2n} q^{\frac{1}{2}n(5n+1)}$$
$$+ \frac{1}{1-xq} \sum_{n=1}^{\infty} (-1)^n \frac{(1-xq)\dots(1-xq^{n+1})}{(1-q)\dots(1-q^n)} x^{2n+1} q^{\frac{1}{2}n(5n+1)+3n+1},$$

where the last step has been achieved by rewriting

$$(1 - x^2q^{4n+2}) - q^n(1 - xq^{2n+1}) = (1 - q^n) + xq^{3n+1}(1 - xq^{n+1}).$$

Now we separate the first sum in the first infinite series on the right, re-index, and deduce that

$$\begin{aligned}\mathcal{H}(x) &= xq(1-xq^2)\\ &\quad + \frac{1}{1-xq}\sum_{n=1}^{\infty}\frac{(-1)^n(1-xq)\dots(1-xq^{n+1})}{(1-q)\dots(1-q^n)}\left[x^{2n+1}q^{\frac{n(5n+1)}{2}+3n+1} - x^{2n+2}q^{\frac{(n+1)(5n+6)}{2}}\right]\\ &= xq(1-xq^2)\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(1-xq^3)\dots(1-xq^{n+1})}{(1-q)\dots(1-q^n)}x^{2n}q^{\frac{n(5n+1)}{2}+3n}(1-xq^{2n+2})\right)\\ &= xq(1-xq^2)\mathcal{G}(xq^2).\end{aligned}$$

■

Lemma 5.6. *For $\mathcal{G}(x)$ as in Lemma 5.3, if $|x| \leq 1$, we have*

$$\mathcal{G}(x)\prod_{n=1}^{\infty}(1-xq^n)^{-1} = 1 + \sum_{n=1}^{\infty}\frac{x^n q^{n^2}}{(1-q)\dots(1-q^n)}.$$

Proof. Let us denote the left side by $\mathcal{F}(x)$. Then Lemma 5.5 takes the form

$$\mathcal{F}(x) = \mathcal{F}(xq) + xq\mathcal{F}(xq^2).$$

Then we find that

$$\mathcal{F}(x) = 1 + \sum_{n=1}^{\infty}\frac{x^n q^{n^2}}{(1-q)\dots(1-q^n)}.$$

We omit the last justification because the reader will have no difficulty in supplying it inasmuch as it is completely analogous to the proof of Lemma 4.3 detailed above. ■

We can now prove Theorem 5.1.

Proof. By putting $x = 1$ and $x = q$ successively in Lemma 5.6, we find, using the Rogers-Ramanujan functions defined in Definition 2.5,

$$\mathcal{G}(1)\prod_{n=1}^{\infty}(1-q^n)^{-1} = G(q), \qquad \mathcal{G}(q)\prod_{n=1}^{\infty}(1-q^{n+1})^{-1} = H(q).$$

But now, using the definition of $\mathcal{G}$, given in Lemma 5.3, and Corollaries 4.6 and 4.7 we find that

$$\mathcal{G}(1) = 1+\sum_{n=1}^{\infty}(-1)^n q^{\frac{n(5n-1)}{2}}(1+q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(5n-1)}{2}} = \prod_{n=1}^{\infty}(1-q^{5n-3})(1-q^{5n-2})(1-q^{5n}),$$

and that

$$\begin{aligned}(1-q)G(q) &= 1+\sum_{n=1}^{\infty}(-1)^n q^{\frac{n(5n+3)}{2}} - q + \sum_{n=1}^{\infty}(-1)^{n+1}q^{\frac{n(5n+3)}{2}}q^{2n+1}\\ &= 1+\sum_{n=1}^{\infty}(-1)^n q^{\frac{n(5n+3)}{2}} + \sum_{n=1}^{\infty}(-1)^n q^{\frac{n(5n-3)}{2}} = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(5n-3)}{2}}\\ &= \prod_{n=1}^{\infty}(1-q^{5n-4})(1-q^{5n-1})(1-q^{5n}).\end{aligned}$$

This means that

$$G(q) = \prod_{n=1}^{\infty}(1-q^{5n-4})(1-q^{5n-1}), \qquad H(q) = (1-q^{5n-2})(1-q^{5n-3}).$$

■

Using the Rogers-Ramanujan identities, we are now able to express $R(q)$ as an infinite product.

Theorem 5.7.

$$R(q) = q^{\frac{1}{5}} \prod_{n=1}^{\infty} \frac{(1-q^{5n-4})(1-q^{5n-1})}{(1-q^{5n-2})(1-q^{5n-3})}.$$

Proof. This is an immediate consequence of Corollary 3.5 and Theorem 5.1. ■

6. Theta Function Transformations.

The importance of the theta function transformations cannot be overstated. To give but one example of their depth, we note that Riemann made use of the first result in Theorem 6.1 in order to give a second elegant proof of the functional equation of the Riemann zeta function in his epoch-making memoir published in 1859, where the Riemann hypothesis was first stated, a transcription of which can be found in [Rie05]. As we will see later on, it is the theta function transformations which will make the actual closed form evaluation of $R(e^{-2\pi})$ possible in this paper.

Theorem 6.1 (Theta Function Transformations). *For the theta functions defined in 2.1,*

$$\vartheta_3(0,q) = \frac{1}{\sqrt{-i\tau}}\vartheta_3(0,q').$$
$$\vartheta_4(0,q) = \frac{1}{\sqrt{-i\tau}}\vartheta_2(0,q').$$
$$\vartheta_2(0,q) = \frac{1}{\sqrt{-i\tau}}\vartheta_4(0,q').$$

To prove these results, our chief instrument will be the Poisson summation formula which we will prove shortly. This result relates the sum of a function at integer points to the sum of its Fourier transform at integer points. When dealing with Fourier series we often want to do things such as differentiating infinite series termwise and interchanging summation and integration signs. Accordingly, we want our functions to be as 'nice' as possible in order for these operations to be justified. The technical term for 'nice' function in this case is 'Schwartz function'.

Lemma 6.2. *Let $f : \mathbb{R} \to \mathbb{C}$ be an infinitely differentiable function such that $f^{(n)}(x) = o(|x|^{-N})$ for all nonnegative integers n and N as $x \to \pm\infty$ (i.e. f rapidly decays to 0 as $x \to \pm\infty$). We call f a Schwarz function. Define the Fourier transform of f as*

$$\hat{f}(x) := \int_{-\infty}^{\infty} f(y)e^{-2\pi i y x}dy.$$

Then

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$$

for all $x \in \mathbb{R}$.

Proof. For all $x \in \mathbb{R}$, define $F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$. Since f is Schwartz, the infinite series $\sum_{n=-\infty}^{\infty} f^{(m)}(x+n)$ are absolutely convergent for all non-negative integers m for all $x \in \mathbb{R}$, and $F(x+1) = F(x)$. Thus $F(x)$ can be expanded into a Fourier series

$$F(x) = \sum_{n=-\infty}^{\infty} c(n) e^{2n\pi i x}$$

where

$$c(n) = \int_{-1/2}^{1/2} F(y) e^{-2n\pi i y} dy = \int_{0}^{1} F(y) e^{-2n\pi i y} dy.$$

Now since f is Schwartz, therefore $\sum_{n=-\infty}^{\infty} f(x+n)$ converges uniformly in the finite interval $(0,1)$ and it may therefore be integrated term-wise there. Hence,

$$c(n) = \sum_{m=-\infty}^{\infty} \int_{0}^{1} f(y+m) e^{-2n\pi i y} dy = \int_{-\infty}^{\infty} f(y) e^{-2n\pi i y} dy = \hat{f}(n).$$

Therefore

$$F(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2n\pi i x}.$$

■

The Poisson summation formula, stated in the next theorem, follows easily from the previous lemma.

Theorem 6.3 (Poisson Summation Formula). *If* $f : \mathbb{R} \to \mathbb{C}$ *is a Schwartz function, then*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

Proof. This follows by putting $x = 0$ in the previous lemma. ■

Before coming to the proof of the theta function transformations, we will fist need a lemma.

Lemma 6.4. *Define the gaussian function* $f(x) = e^{-\pi x^2}$ *for all* $x \in \mathbb{R}$*.Then* $f(x) = \hat{f}(x)$.

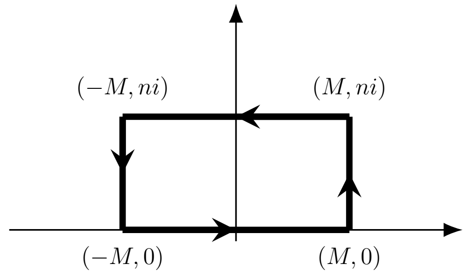
Proof. We have

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-\pi y^2} e^{-2\pi i y x} dy.$$

Let $n \in \mathbb{R}_{>0}$ and let $M > 0$ arbitrarily. Let R denote the positive oriented rectangle with vertices at the points $(-M,0), (M,0), (M,ni)$ and $(-M,ni)$ (see Figure 1). Then, since e^{-z^2} is an entire function of z,

$$\int_R e^{-z^2-n^2} dz = 0$$

Figure 1. The positively oriented rectangular contour R.



by the Cauchy-Goursat theorem. A straightforward application of the ML-inequality reveals that the integral vanishes over the vertical sides of R as $M \to \infty$. Therefore

$$\lim_{M\to\infty} \int_R e^{-z^2-n^2} dz = \lim_{M\to\infty} \int_{-M}^{M} e^{-z^2-n^2} dz + \lim_{M\to\infty} \int_{M}^{-M} e^{-(z+ni)^2-n^2} dz$$

so that

$$\int_{-\infty}^{\infty} e^{-z^2-2nzi} dz = \int_{-\infty}^{\infty} e^{-z^2-n^2} dz.$$

If we changed z to $-z$ we also find that

$$\int_{-\infty}^{\infty} e^{-z^2+2nzi} dz = \int_{-\infty}^{\infty} e^{-z^2-n^2} dz.$$

By combining these two results we deduce that, for all $n \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{-z^2} \cos(2nz) dz = \int_{-\infty}^{\infty} e^{-z^2-n^2} dz = e^{-n^2}\sqrt{\pi},$$

where we have made use of the famous Gaussian integral

$$\int_{\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}.$$

But now

$$\int_{-\infty}^{\infty} e^{-z^2} \sin(2nz) dz = 0$$

since the integrand is odd. This means that

$$\int_{-\infty}^{\infty} e^{-y^2}(\cos(2ny) + i\sin(2ny)) dy = \int_{-\infty}^{\infty} e^{-y^2} e^{-2nyi} dy = \sqrt{\pi} e^{-n^2}.$$

By changing y into $\sqrt{\pi}y$ and n into $\sqrt{\pi}x$, we deduce that

$$\int_{-\infty}^{\infty} e^{-\pi y^2} e^{2x\pi yi} dy = \hat{f}(x) = e^{-\pi x^2} = f(x)$$

for all $x \in \mathbb{R}$. ∎

Corollary 6.5. *For all $x \in \mathbb{R}$ and all $t > 0$, we define $f_t(x) = e^{-\pi t x^2}$. Then $\hat{f}_t(x) = t^{-\frac{1}{2}} e^{-\frac{\pi x^2}{t}}$.*

Proof. From the previous lemma we have

$$\int_{-\infty}^{\infty} e^{-\pi y^2} e^{2\pi i x y} dy = e^{-\pi x^2}.$$

The required result follows upon substituting $yt^{\frac{1}{2}}$ for y and $xt^{-\frac{1}{2}}$ for x. ■

We can now finally prove the theta function transformations in Theorem 6.1.

Proof. First of all, we note that the second and third equations, respectively, follow from the first by changing z into $z + \frac{1}{2}\pi$ and $z + \frac{1}{2}\pi\tau$, using Corollary 2.4. It therefore suffices to prove only the first transformation.
Applying the Poisson summation formula of Theorem 6.3 to the Schwartz function $f_t(x) = e^{-\pi t x^2}$ for all $x \in \mathbb{R}$ and $t > 0$ from the previous Corollary, and using its Fourier Transform found there, we find that

$$\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} = t^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{t}}.$$

This is precisely the result which we wish prove but only for all τ on the positive imaginary axis, which is a set of points in the open half upper τ-plane with limit point. But both sides extend to a holomorphic function on the entire upper half τ-plane, namely the right side to $\vartheta_3(0, q)$ and the left side to $(-i\tau)^{\frac{1}{2}}\vartheta_3(0, q')$. Since they agree on a set with limit point they have the same analytic continuation to the upper half-plane and we thence conclude that

$$\vartheta_3(0, q) = (-i\tau)^{\frac{1}{2}}\vartheta_3(0, q')$$

for all $\Im(\tau) > 0$. ■

We will now utilise the theta function transformations to achieve our next goal for this section, which is to prove the relation (1.6) as our next theorem.

Theorem 6.6.

$$e^{\frac{\pi i}{2}\left(\tau + \frac{1}{\tau}\right)} \prod_{n=1}^{\infty} \left(\frac{1 - e^{2n\pi i \tau}}{1 - e^{-\frac{2n\pi i}{\tau}}} \right)^6 = \frac{1}{i\tau^3}.$$

Proof. By putting $z = 0$ in Corollary 4.5, and multiplying the results together, we find that

$$\vartheta_2(0, q)\vartheta_3(0, q)\vartheta_4(0, q) = 2q^{\frac{1}{4}} \prod_{n=1}^{\infty} (1 - q^{2n})^3,$$

so that

$$\prod_{n=1}^{\infty} (1 - q^{2n})^6 = \frac{1}{4} q^{-\frac{1}{2}} \vartheta_2^2(0, q)\vartheta_3^2(0, q)\vartheta_4^2(0, q).$$

Therefore, using the theta function transformations of Theorem 6.1, we have

$$\begin{aligned}
\prod_{n=1}^{\infty} \left(\frac{1 - q^{2n}}{1 - q'^{2n}} \right)^6 &= \frac{q^{-\frac{1}{2}}}{q'^{-\frac{1}{2}}} \cdot \frac{\vartheta_2^2(0, q)}{\vartheta_4^2(0, q')} \cdot \frac{\vartheta_3^2(0, q)}{\vartheta_3^2(0, q')} \cdot \frac{\vartheta_4^2(0, q)}{\vartheta_2^2(0, q')} \\
&= \frac{q^{-\frac{1}{2}}}{q'^{-\frac{1}{2}}} \cdot \frac{1}{i\tau^3}.
\end{aligned}$$

■

The relevant particular case of the previous theorem for this paper is (1.5), which is proved in the next corollary.

Corollary 6.7.

$$e^{\frac{2\pi}{5}} \prod_{n=1}^{\infty} \left(\frac{1 - e^{-\frac{2n\pi}{5}}}{1 - e^{-10n\pi}} \right) = \sqrt{5}.$$

Proof. This is an immediate consequence of putting $\tau = i/5$ in Theorem 6.6 and taking the sixth root of both sides, remembering that the result for each side should be a positive number. ■

7. Ramanujan's first non-elementary evaluation of $R(q)$.

We have now reached the last stages of our little adventure. We start we a simple lemma.

Lemma 7.1.

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} = 0.$$

Proof. By absolute convergence, the sum is unchanged when we replace n by $-n-1$. But in so doing, the sum becomes $-\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(n+1)}{2}}$, which is the negative of the original series. We conclude that the latter equals zero. ■

In our next theorem, we prove (1.3), one of the most famous and useful results about $R(q)$.

Theorem 7.2.

$$\frac{1}{R(q)} - 1 - R(q) = q^{-\frac{1}{5}} \prod_{n=1}^{\infty} \left(\frac{1 - q^{\frac{n}{5}}}{1 - q^{5n}} \right).$$

Proof.

$$\prod_{n=1}^{\infty} \left(1 - q^{\frac{n}{5}}\right)$$

$$= \frac{\left\{\prod_{n=0}^{\infty} \left(1 - q^{\frac{5n+1}{5}}\right)\left(1 - q^{\frac{5n+4}{5}}\right)\left(1 - q^{\frac{5n+5}{5}}\right)\right\}\left\{\prod_{n=0}^{\infty} \left(1 - q^{\frac{5n+2}{5}}\right)\left(1 - q^{\frac{5n+3}{5}}\right)\left(1 - q^{\frac{5n+5}{5}}\right)\right\}}{\prod_{n=0}^{\infty}\left(1 - q^{\frac{5n+5}{5}}\right)}$$

$$= \frac{\left\{\sum_{r=-\infty}^{\infty}(-1)^r q^{\frac{r(5r-3)}{10}}\right\}\left\{\sum_{s=-\infty}^{\infty}(-1)^s q^{\frac{s(5s-1)}{10}}\right\}}{\prod_{n=0}^{\infty}(1 - q^{n+1})},$$

where we have made use of Corollaries 4.6 and 4.7. We rewrite the last formula as

$$\frac{\left\{\sum_{r=-\infty}^{\infty}(-1)^r a^r \cdot q^{\frac{r^2-r}{2}}\right\}\left\{\sum_{s=-\infty}^{\infty}(-1)^s a^{2s} \cdot q^{\frac{s^2-s}{2}}\right\}}{\prod_{n=0}^{\infty}(1 - q^{n+1})},$$

where we have put $a = q^{\frac{1}{5}}$. We can put the last formula equal to

$$\sum_{n=-\infty}^{\infty} a^n c_n(q),$$

where

$$c_n(q) = \prod_{k=0}^{\infty}(1-q^{k+1})^{-1} \cdot \sum_{r+2s=n} (-1)^{r+s} q^{\frac{r^2-r+s^2-s}{2}}.$$

We will now evaluate $c_{5n}(q), c_{5n+1}(q), c_{5n+2}(q), c_{5n+3}(q)$ and $c_{5n-1}(q)$ successively.
Putting $r = n-2t$ and $s = 2n+t$, so that $r+2s = 5n$, and summing over all t, we find

$$\begin{aligned} c_{5n}(q) &= \prod_{k=0}^{\infty}(1-q^{k+1})^{-1} \cdot \sum_{t=-\infty}^{\infty} (-1)^{n+t} q^{\frac{[(n-2t)^2-(n-2t)+(2n+t)^2-(2n+t)]}{2}} \\ &= (-1)^n q^{\frac{5n^2-3n}{2}} \cdot \prod_{k=0}^{\infty}(1-q^{k+1})^{-1} \cdot \sum_{t=-\infty}^{\infty} (-1)^t q^{\frac{5t^2+t}{2}} \\ &= \frac{(-1)^n q^{\frac{5n^2-3n}{2}}}{\prod_{k=0}^{\infty}(1-q^{5k+1})(1-q^{5k+4})}, \end{aligned}$$

where we have used Corollary 4.6.
If we put $r = n+1-2t$ and $s = 2n+t$, so that $r+2s = 5n+1$, and we sum over all t, we find

$$\begin{aligned} c_{5n+1}(q) &= \prod_{k=0}^{\infty}(1-q^{k+1})^{-1} \cdot \sum_{t=-\infty}^{\infty} (-1)^{n+t+1} q^{\frac{[(n+1-2t)^2-(n+1-2t)+(2n+t)^2-(2n+t)]}{2}} \\ &= (-1)^{n+1} q^{\frac{5n^2-n}{2}} \cdot \prod_{k=0}^{\infty}(1-q^{k+1})^{-1} \cdot \sum_{t=-\infty}^{\infty} (-1)^t q^{\frac{5t^2-3t}{2}} \\ &= -\frac{(-1)^n q^{\frac{5n^2-n}{2}}}{\prod_{k=0}^{\infty}(1-q^{5k+2})(1-q^{5k+3})}, \end{aligned}$$

where now we have used Corollary 4.7.
Similarly, we evaluate $c_{5n+2}(q)$ and $c_{5n+3}(q)$, respectively, by putting $(r,s) = (n-2t, 2n+1+t)$ and $(r,s) = (n+1-2t, 2n+1+t)$, summing over all t and using Corollaries 4.7 and 4.6. In so doing we obtain

$$c_{5n+2}(q) = -\frac{(-1)^n q^{\frac{5n^2+n}{2}}}{\prod_{k=0}^{\infty}(1-q^{5k+2})(1-q^{5k+3})}, \qquad c_{5n+3}(q) = \frac{(-1)^n q^{\frac{5n^2+3n}{2}}}{\prod_{k=0}^{\infty}(1-q^{5k+1})(1-q^{5k+4})}.$$

Lastly, we put $r = n-1-2t$ and $s = 2n+t$, so that $r+2s = 5n-1$, and by summing over all t we obtain

$$\begin{aligned} c_{5n-1}(q) &= \prod_{k=0}^{\infty}(1-q^{k+1})^{-1} \cdot \sum_{t=-\infty}^{\infty} (-1)^{n+t+1} q^{\frac{[(n-1-2t)^2-(n-1-2t)+(2n+t)^2-(2n+t)]}{2}} \\ &= (-1)^{n+1} q^{\frac{5n^2-5n+2}{2}} \cdot \prod_{k=0}^{\infty}(1-q^{k+1})^{-1} \cdot \sum_{t=-\infty}^{\infty} (-1)^t q^{\frac{5t^2+5t}{2}} = 0, \end{aligned}$$

by Lemma 7.1. From these evaluations it now follows that

$$\prod_{n=1}^{\infty}(1-q^{\frac{n}{5}}) = \frac{\sum_{n=-\infty}^{\infty}(-1)^n a^{5n} q^{\frac{5n^2-3n}{2}}}{\prod_{k=0}^{\infty}(1-q^{5k+1})(1-q^{5k+4})} - \frac{\sum_{n=-\infty}^{\infty}(-1)^n a^{5n+1} q^{\frac{5n^2-n}{2}}}{\prod_{k=0}^{\infty}(1-q^{5k+2})(1-q^{5k+3})}$$
$$- \frac{\sum_{n=-\infty}^{\infty}(-1)^n a^{5n+2} q^{\frac{5n^2+n}{2}}}{\prod_{k=0}^{\infty}(1-q^{5k+2})(1-q^{5k+3})} + \frac{\sum_{n=-\infty}^{\infty}(-1)^n a^{5n+3} q^{\frac{5n^2+3n}{2}}}{\prod_{k=0}^{\infty}(1-q^{5k+1})(1-q^{5k+4})}.$$

Recalling that $a = q^{\frac{1}{5}}$, we see that the fourth summand on the right side is zero because the numerator has the factor $\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5n^2+5n}{2}}$ which equals zero by Lemma 7.1. When now we apply Corollaries 4.6 and 4.7 on the right side, it becomes

$$\prod_{n=0}^{\infty}(1-q^{5n+5})\left\{\prod_{k=0}^{\infty}\frac{(1-q^{5k+2})(1-q^{5k+3})}{(1-q^{5k+1})(1-q^{5k+4})} - q^{\frac{1}{5}} - q^{\frac{2}{5}}\prod_{k=0}^{\infty}\frac{(1-q^{5k+1})(1-q^{5k+4})}{(1-q^{5k+2})(1-q^{5k+3})}\right\}.$$

By using Theorem 5.7, we deduce that

$$\prod_{n=1}^{\infty}(1-q^{\frac{n}{5}}) = q^{\frac{1}{5}}\left\{\frac{1}{R(q)} - 1 - R(q)\right\}\prod_{n=1}^{\infty}(1-q^{5n}).$$

■

At last, we arrive at Ramanujan's evaluation of $R(q)$.

Theorem 7.3.

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}$$

Proof. Putting $q = e^{-2\pi}$ in Theorem 7.2 and using Corollary 6.7, we obtain

$$\frac{1}{R(e^{-2\pi})} - 1 - R(e^{-2\pi}) = e^{\frac{2\pi}{5}}\prod_{n=1}^{\infty}\left(\frac{1-e^{-\frac{2n\pi}{5}}}{1-e^{-10n\pi}}\right) = \sqrt{5}.$$

The evaluation follows by solving the quadratic equation and noting that $R(e^{-2\pi}) > 0$. ■

8. Further Results.

There is a vast amount of beautiful mathematics surrounding the Rogers-Ramanujan continued fraction. The particular evaluation proved in this paper is fascinating, although it is only the tip of the iceberg. With only a little additional work, the results in this paper can be used to established the evaluation

$$-R(-e^{-\pi}) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2},$$

as well as the modular equation of degree 5 in the next theorem:

Theorem 8.1. *If $v = R(q)$ and $u = R(q^5)$, then*

$$v^5 = u\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}.$$

Both of the previous results were in Ramanujan's first letter to Hardy. For proofs the reader may consult [Wat29b]. The proof of another evaluation of Ramanujan, namely

$$R(e^{-2\pi\sqrt{5}}) = \frac{\sqrt{5}}{1+\left(5^{\frac{3}{4}}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{5}{2}}-1\right)^{\frac{1}{5}}} - \frac{\sqrt{5}+1}{2},$$

can be found in [Wat29a], where the author also proves another remarkable theorem of Ramanujan, namely:

Theorem 8.2. *If $\alpha\beta = \pi^2$, then*

$$\left\{\frac{\sqrt{5}+1}{2} + R(e^{-2\alpha})\right\}\left\{\frac{\sqrt{5}+1}{2} + R(e^{-2\beta})\right\} = \frac{5+\sqrt{5}}{2},$$

with which one can obtain two closed-form evaluations of $R(q)$ from only one. One single method is not sufficient in order to obtain other beautiful closed-form evaluations of $R(q)$. For many such evaluations, as well as other wonderful results about $R(q)$, the reader is strongly encouraged to study [BA05], in which an extensive list of references to the literature is also to be found. For a particularly amazing theorem about closed-form evaluations of $R(e^{\pi\sqrt{n}})$ whenever n is a positive rational number, the reader is referred to [BCZ96, Theorem 6.2]. The Rogers-Ramanujan identities in Theorem 5.1 also have a vast amount of literature about them.The reader interested in studying more about them is encouraged to consult Berndt's commentary on the paper [RR19] of Rogers and Ramanujan, found in [Ram00, pp. 375-378].

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