Motivating de Rham cohomology

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Motivation

Motivating the talk

I wish to give my audience a glimpse into why I found de Rham cohomology, its consequences and prerequisite mechanisms so exciting. I hope to motivate the audience enough to where they will engage with de Rham cohomology independently.

Motivating de Rham cohomology

We motivate de Rham cohomology simply as a tool to detect holes in the domain. Recall the fact from multivariable calculus that a vector field $\mathbf{F}$ with curl $\mathbf{F} = 0$ is the gradient of a function. Turns out, this fails if the simply connected domain has a hole.

Motivation - Example

Example

Example If $U = \mathbb{R}^3 \backslash \{z\text{-axis}\}$, and

$$\mathbf{F} = \left\langle \frac{-y^2}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right\rangle.$$

To show that curl $\mathbf{F} = 0$, let $\mathbf{F} = \langle P, Q, R \rangle$. Then,

$$P = \frac{-y}{x^2+y^2},\ Q = \frac{x}{x^2+y^2},\ R = 0.$$

By curl $\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$, the first, second and third components evaluate to 0.

Thus, an integral about a loop, must be zero. Take $s = (\cos t, \sin t)$, the unit circle. Then (by parametrisation and the dot product),

$$\int_s \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(s(t)) \cdot s'(t) dt = \int_0^{2\pi} 1\ dt = 2\pi - 0 = 2\pi \neq 0.$$

Preliminary Constructions

Smooth Functions

A function is considered smooth or C^∞ if

$$\frac{\partial f^k}{\partial x^1 \partial x^2 \ldots \partial x^k}$$

is defined and continuous over a point $x \in \mathbb{R}^k$.

Tangent Spaces

A tangent space over a point $p \in \mathbb{R}^k$, denoted $T_p(U)$, for U is an open set containing p, is the vector space of all vectors tangent to each component of $p = (p_1, p_2, \ldots, p_k)$.

Vector Fields in $\mathbb{R}^k$

A vector field in $\mathbb{R}^k$ is a function that maps one vector from the tangent space $T_p(U)$ to p.

Exterior Algebra - Dual Space

Dual Space

The space $\text{Hom}(V, \mathbb{R})$ is the set of all linear maps from $V \to \mathbb{R}$, known as the dual space of V, denoted V^V. Its elements are called 1-covectors.

We care about dual spaces because it allows for defining *directional derivatives* with respect to the elements of the vector space.
We define the directional derivative of a function $f \in C^\infty$ about a neighborhood of a point p at p to be

$$D_v = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}\bigg|_p ,$$

taking $v_i \in v \in V$, thus assigning a 'direction' to each partial derivative component.

Exterior Algebra - Tensors

k-tensor space

Consider the cartesian product of vector space

$$V^k = \underbrace{V \times V \times \cdots \times V}_{k}.$$

Trivially, this is a vector space. Thus, the dual space of V^k is the set of all k-tensors, denoted $L_k(V)$, is the vector space of all maps $f : V^k \to \mathbb{R}$, satisfying the multilinearity property

$$f(\dots, av + bw, \dots) = af(\dots, v, \dots) + bf(\dots, w, \dots).$$

k-tensors are multilinear maps. Intuitively, this means that each argument is linear when all others are fixed.

We also have an operation that maps $A_k(U)$ and $A_l(U)$ to $A_{k+l}(U)$. We will define it no further.

$$\wedge : A_k(U) \times A_l(U) \to A_{k+l}(U).$$

Differential Forms

Differential Forms

A differential form can be understood as a covector field; a vector field which maps 1-covectors from the cotangent space $T_p^*(U)$ to p.

Differential 1-forms and the differential

A covector field mapping 1-covectors from $T_P^*(U)$ to p. A nice example is the differentials, dx, dy, dz in $\mathbb{R}^3$. Thus, 1-forms or differentials motivate calculus in $\mathbb{R}^3$.

Differential k-forms

We can define the differential k-form about a point $p \in \mathbb{R}^n$ to be

$$\omega_p : T_p(\mathbb{R}^n) \times T_p(\mathbb{R}^n) \times \cdots \times T_p(\mathbb{R}^n) \to \mathbb{R}.$$

See how its similar to a k-tensor, but for tangent spaces instead.

Here's a cool geometric perspective: A differential k-form returns the oriented volume of the dim k parallelepiped spanned by k-tangent vectors. Use this for intuition.

Exact and Closed Forms, the Exterior Derivative

Exterior Derivative

We first note that

$$\wedge : \Omega^k(\mathbb{R}^n) \times \Omega^l(\mathbb{R}^n) \to \Omega^{k+l}(\mathbb{R}^n),$$

when the wedge product is defined over differential forms. Take this on faith. We now say that the exterior derivative is a map

$$d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n), \quad d_e : \omega \to d\omega.$$

Closed Forms

A closed form is a k-form such that its differential $d\omega = 0$.

Exact Forms

An exact form is a k-form ω such that $\exists \tau$, a $k-1$-form such that $\omega = d\tau$.

Exact Sequences

A sequence of homomorphisms on vector spaces

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if $\operatorname{Im} f = \ker g$. The logic behind this is that by exact forms $\omega = d\tau$, and they are closed meaning $d\omega = 0$. This means $d\tau \to \omega \to d\omega$ is analogous to this sequence. The rest is self-explanatory. This blows up into large sequences, where we cannot consider the first and last terms as exact because then it would blow up to infinity.

$$A^0 \xrightarrow{f^0} A^1 \xrightarrow{f^1} A^2 \xrightarrow{f^2} \dots \xrightarrow{f^{n-1}} A^n.$$

Cochain Complex

Defining the Cochain Complex

A cochain complex $\mathcal{C}$ is a collection of vector spaces $\{C^k\}_{k\in\mathbb{Z}}$ together with a sequence of linear maps $d_{\cdot} C^k \to C^{k+1}$,

$$\dots \xrightarrow{d_{-2}} C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \dots$$

such that,

$$d_k \circ d_{k-1} = 0, \ \forall k.$$

Obscure property? No. Here's an original explanation: If we map d_{k-1} to the next vector space, we receive the image of it. now if we apply d_k to that image, then we're effectively applying d_k to the kernel of d_k, which is 0, since $\operatorname{Im} d_{k-1} \subset \ker d_k$. However, this creates a circular argument with $\operatorname{Im} (d_{k-1}) \subset \ker d_k$, with both proving each other (they are equivalent statements). This is just a part of the machinery of cochain complexes.

de Rham complex

The de Rham complex is a cochain complex of $\Omega^*(M)$. In this case, the vector spaces are the vector spaces $\Omega^k(\mathbb{R}^n)$, and the homomorphism is the exterior derivative d, and the property $d \circ d = 0$.

$$0 \to \Omega^0(\mathbb{R}^n) \xrightarrow{d} \Omega^1(\mathbb{R}^n) \xrightarrow{d} \Omega^2(\mathbb{R}^n) \xrightarrow{d} \dots \xrightarrow{d} \Omega^k(\mathbb{R}^n) \to \dots$$

A property of the de Rham complex is that the image of each homomorphism is contained in the kernel of the next, by the definition of a cochain complex. This means that $\text{Im } d \in \ker d$, however the converse may not be true. Recall that $d^2 = 0$, $d(d\tau) = 0$ thus every exact form is closed, but not the converse. This is very similar to closed and exact forms.

An essential idea

The idea of exactness is associated with the 'consistency and stability' of the de Rham complex without holes. It is exact when it has no holes, thus the *de Rham cohomology* is 0 by its quotient definition. When there is a hole, the de Rham complex is non-exact.

Detecting holes I - Stokes' Theorem & the geometric interpretation

Recall the generalised Stoke's Theorem, and the the condition of the de Rham complex where $d^2 = 0$ for d is the exterior derivative defining $k+1$-forms from k-forms,

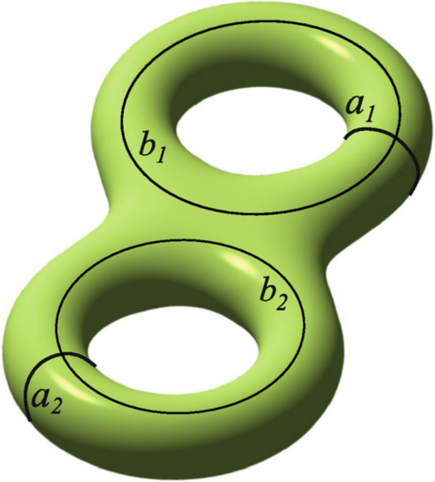
$$\int_D d\omega = \int_{\partial D} \omega$$

for ω is a k-form. Recall the definition of an exact form τ such that $\exists \sigma$, a $k-1$-form such that $\tau = d\sigma$, we can define about the boundary of a disk D, denoted by ∂D,

$$\int_{\partial D} d\sigma = \int_{\partial(\partial D)} \tau = \int_0 \tau = 0.$$

Think of exterior derivative as a map between consecutive-dimensional surfaces; a form on a line (a boundary) maps to a form on a surface (the disk enclosed by the boundary). So we use the exterior derivative property on the boundary vs. the plane D, following $d^2 = 0$, thus we can detect whether there is a hole if the period of a differential form is 0.

Example



Detecting holes II - the de Rham cohomology

de Rham cohomology

A de Rham cohomology is the quotient vector space of closed and exact forms

$$H^k(M) = \frac{Z^k(M)}{B^k(M)},$$

where $Z^k(M)$ is the set of all closed forms and $B^k(M)$ is the set of all exact forms. Alternatively, by the containment of $\text{Im } d \in \ker d$,

$$H^k(M) = \frac{\ker d}{\text{Im } d}.$$

The idea behind this leads back to closed and exact forms; all exact forms are closed, but there can exist non-0-closed forms ω that are not exact. What this means is that $d\omega = 0$, but $\omega \neq d\tau$, for any $k-1$-form τ. So 'zero-ness' arbitrarily stems from ω, which seems odd.
When $H^k(M) \neq 0$, the quotient does not 'cancel out'. Rather it becomes 0, indicating zero differences in the two sets.

Further Reading

My paper!

I have included proofs and a lot of juicy mathematics which motivates most of what may seem oddly developed in this talk. It's a great complement to today's content.

In all seriousness though, go read Tu's 'Introduction to Manifolds' followed by Bott & Tu's 'Differential forms in Algebraic Topology' for a rigorous yet intuitive introduction to de Rham cohomology.