CONSTRUCTING DE RHAM COHOMOLOGY

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ABSTRACT. The function of cohomology in algebraic topology, and its extensions in an array of mathematics, calls upon the mathematics enthusiast to be astute in its general principles. However, it may come at a level of abstraction and maturity for which many may not be suited. This paper aims to introduce de Rham cohomology, a crucial subsection of cohomology, both intuitively and rigorously, stemming from the contents of elementary calculus. Building from its foundations in exterior algebra, we will define differential forms and construct smooth manifolds to explore the detection of 'holes' both geometrically, using the de Rham complex and the generalised Stokes' Theorem, as well as the de Rham cohomology. The paper will conclude with the basic mechanisms of de Rham cohomology.

1. Introduction

I was motivated to write this paper to offer a concrete and multifaceted pathway to introduce de Rham cohomology in the best way by linking its construction to different subfields of mathematics. Thus, this paper adopts different styles of approach to create a unique pedagogy to de Rham cohomology.

First, we would like to set up the structure of de Rham cohomology to motivate its construction.

de Rham cohomology is the study of how *cohomologies* as *quotient vector spaces*, labeled $H^k(M)$, defined over a *smooth manifold* M, single-out the topological properties, like holes, in the manifold with properties of *differential forms* or the *exterior derivative* in the *de Rham complex* defined over M. The de Rham cohomology $H^k(M)$ takes the expression

$$H^k(M) = \frac{Z^k(M)}{B^k(M)} = \frac{\ker d}{\operatorname{im} d},$$

for $Z^k(M)$ is the set of all *closed k-forms* on M, and $B^k(M)$ is the set of all *exact k-forms* on M. d is the exterior derivative as the linear map between $\Omega^*(M)$, vector spaces of differential forms over M where $* = 1, 2, 3, \dots$, on the de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \dots.$$

Everything defined above in italics is the culmination of exterior algebra, differential forms, and manifold theory. While it may seem like technical jargon right now, its beauty in the parallels of its structure is revealed from a rigorous and intuitive understanding of its components. Now, we would like to motivate an example of detecting holes; a key component of de Rham cohomology.

Recall the fact from multivariable calculus,

Remark 1.1. A vector field $\mathbf{F}$ with curl $\mathbf{F} = 0$ is the gradient of a scalar function in a simply connected domain.

Definition 1.2. A simply connected domain is one where every closed loop can be contracted to a point without leaving the domain

Thus, a domain with a 'hole' is not simply connected. We can see that the fact fails for an unconnected domain.

Example. [5] Let $U = \mathbb{R}^3 \backslash \{z\text{-axis}\}$, and a vector field $\mathbf{F}$ as

$$\mathbf{F} = \left\langle \frac{-y^2}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right\rangle.$$

We must first show that curl $\mathbf{F} = 0$, we define $\mathbf{F}$ component wise. Let $\mathbf{F} = \langle P, Q, R \rangle$, where

$$P = \frac{-y^2}{x^2+y^2},\ Q = \frac{x}{x^2+y^2},\ R = 0.$$

For curl $\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$, we need to evaluate each component.

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) = \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2} \right) \right) = (0-0) = 0, \tag{1.1}$$

$$\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) = \left(\frac{\partial}{\partial z}\left(\frac{-y^2}{x^2+y^2} \right) - \frac{\partial}{\partial x}(0) \right) = (0-0) = 0, \tag{1.2}$$

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} \right) - \frac{\partial}{\partial y}\left(\frac{-y^2}{x^2+y^2} \right) \right) = 0. \tag{1.3}$$

The last expression was evaluated using the quotient rule. It is long thus has not been included.

Since $\nabla \times \mathbf{F} = (0,0,0)$, curl $\mathbf{F} = 0$. From this we may infer that any integral about a loop must also evaluate to 0.

Claim 1.3. *Any integral about a loop for* $\mathbf{F} = \left\langle \frac{-y^2}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right\rangle$ *must evaluate to* 0 *in a simply connected domain, by Stokes' Theorem.*

The simplest loop for our computation is the unit circle $s = (\cos\theta,\ \sin\theta, 0)$. We then integrate $\mathbf{F}$ over s,

$$\int_s \mathbf{F} \cdot d\mathbf{r}$$

which we parametrise to obtain

$$\int_s \mathbf{F}(s(t)) \cdot s'(t) dt = \int_0^{2\pi} 1\ dt = 2\pi - 0 = 2\pi.$$

This contradicts our claim, which means the domain is not simply connected.

This example highlights how simple integration in $\mathbb{R}^3$ allows us to detect holes. However, a lot of mathematics is centered around different domains, such as smooth manifolds and spaces with different properties. This method is not suitable to detect holes in such spaces, because they may not even be Euclidean in the first place! Thus, we turn our attention to de Rham cohomolgy, which allows us to study the topological properties like holes in a wide variety of spaces.

2. Preliminaries in Euclidean Space [5]

We must first define some constructs which allow us to define calculus, specifically integration, in abstract spaces.

Definition 2.1. A function or its k^{th} derivative is continuous about a structure (point, open set, topological space or entire domain), labeled C^k if its all its derivatives up to the k^{th} index

$$\frac{\partial^k f}{\partial x^{i_1} \dots \partial x^{i_k}}$$

exist and are continuous. A function is smooth or C^∞ over x if all its derivatives exist and are defined about x.

Definition 2.2. A tangent space over a point $p = (p^1, p^2, \dots, p^n) \in \mathbb{R}^n$, denoted $T_p(U)$, for $U \subset \mathbb{R}^n$ is an open subset containing p, is the set (vector space) of all vectors *tangent* to p.

With the vectors in a tangent space, we can define the *directional derivative* of a function $f \in C^\infty$ on p.

Proposition 2.3. *The line through $p = (p^1, \dots, p^n) \in \mathbb{R}^n$ with direction $v = \langle v_1, \dots, v_n \rangle$ has parametrisation*

$$l(t) = (p^1 + tv^1,\ p^2 + tv^2, \dots,\ p^n + tv^n),$$

where its i^{th} component $l^i(t) = (p^i + tv^i)$. If v is a tangent vector, the directional derivative of f in the direction v over p is given by the expression

$$D_v f = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \left.\frac{d}{dt}\right|_{t=0} f(c(t) = \sum_{i=1}^{n} \frac{d\ l^i(0)}{dt}(0)\frac{\partial f(p)}{\partial x^i} = \sum_{i=1}^{n} v^i \frac{\partial f(p)}{\partial x^i}.$$

When considering the partial derivative as an operator on a function, we get

$$D_v = \sum_{i=1}^{n} v^i \left.\frac{\partial}{\partial x^i}\right|_p.$$

We define it as such so tangent vectors $v \in T_p(U)$ can act as operators on functions.

2.1. Germs & Derivations.

Suppose we have pairs $(f, U),\ (g, V)$ where $U,\ V$ are neighborhoods of a point p. Here,

$$f : U \to \mathbb{R}, \quad (2.1)$$

$$g : V \to \mathbb{R} \quad (2.2)$$

are C^∞. The set where both are defined must be $U \cap V$, and to examine how they work as smooth functions locally, if $\exists W \subset U \cap V$ is an open set of p. We say that f and g are equivalent if $f = g$ on W. The equivalence classes of these functions are called their **germs**. The set of all germs over a point $p \in \mathbb{R}^n$ is referred to as $C_p^\infty(\mathbb{R}^n)$, or just C_p^∞. The representative of a germ of f in C_p^∞ is written $[f]$.

We use germs to define the derivation.

Definition 2.4. For each vector v in $T_p(U)$, $U \subset \mathbb{R}^n$ is an open subset containing p, the directional derivative induces a linear map

$$D_v : C_p^\infty \to \mathbb{R}$$

that satisfies the Leibniz rule

$$D_v(fg) = f(p)(D_v g) + (D_v f)g(p)$$

as a result of the properties of partial derivatives. Any linear map satisfying these conditions is called a **derivation** of C_p^∞.

Derivations are crucial in our argument for *differentials of differential forms.*

2.2. Vector Fields.

Definition 2.5. A vector field is a function defined over an open subset U that assigns a tangent vector from each tangent space about $p \in U$, to each point in U.

Since $T_p(U)$ has the partial derivatives basis, we can write a vector field about a point p as

$$X_p = \sum a^i(p) \frac{\partial}{\partial x^i}\bigg|_p.$$

Thus, a vector field about the entire field U would be

$$X = \sum a^i \frac{\partial}{\partial x^i}.$$

A vector field can be thought of as a derivation as well. Its mathematical exposition, however, is unimportant to this papers arguments. Later, the reader may notice the parallels between *covector* fields and vector fields in terms of *differentials* and derivations not explicitly stated.

3. Exterior Algebra of Tensors [5]

Consider a vector space V with basis $e_1, e_2, \ldots, e_n$. Like defined previously, a linear map is a map between two vector spaces satisfying two properties listed above. Here, we can define a new vector space V^* called the **dual space**, which is the set of all maps from $V \to \mathbb{R}$, defined by

$$V^* = \text{Hom}(V, \mathbb{R}).$$

The elements of V^* are called **covectors**, **1-covectors** or **1-tensors**.

Additionally, $\forall v \in V$, we write

$$v = \sum_{i,j} v^i e_j$$

We say $\alpha^1, \alpha^2, \ldots, \alpha^n$ is the basis for V^* (proof is omitted). If the map $\alpha^i(v) = v^i$, gives the index i, we say

$$\alpha^i(e_j) = \delta^i_j = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

We want to prove that $|V| = |V^*|$, thus $|e_n| = |\alpha^n|$. To do this, we can take a map $b^i(v) \in \mathbb{R}$. Thus,

$$a^i(v) = \sum_j b^j(v)\ e_j = \sum_j b^j(v)\delta^i_j = b^i(v).$$

3.1. **Permutations & Tensors.** A permutation is a function which reorders a set. Consider the set $\{1, 2, 3, \ldots, k\}$, and the set of all permutations of these to be S_k.

Definition 3.1. We say a permutation σ is cyclic if $\sigma(1) = 2,\ \sigma(2) = 3, \ldots, \sigma(k-1) = k,\ \sigma(k) = 1$. This means σ shifts elements in a single cycle.

Example. An example of a cyclic permutation is one with mod

$$\sigma(i) = (i + 1) \bmod k.$$

We do not care about any particular σ, but any σ.

The sign of σ denoted $\operatorname{sgn}(\sigma)$ is the sign of the permutation: it is $+1$ if it can be rewritten as a combination of an even number of *transpositions*, and is -1 if it can be rewritten as a combination of an odd number of transpositions.

Definition 3.2. A transposition is a 2-cycle, that is, a cycle of the form $(a\ b)$ that interchanges a and b, leaving all other elements of A fixed.

Definition 3.3. Consider the vector space $V^k = V \times V \times V \times \cdots \times V$ which is a result of the Cartesian product of k vector spaces V. A map from $f : V^k \to \mathbb{R}$ is called a k**-tensor** if:

$$f(\ldots, av + bw, \ldots) = af(\ldots, v, \ldots) + bf(\ldots, w, \ldots).$$

Apart from my personal bias for the word tensor, it is very important in our construction of differential forms. A k-tensor an also be called a multinlinear map of k-arguments. The set of all k-tensors for V^k is $L_k(V)$.

Intuitively, the multilinearity property states that for all k-arguments taken by the k-tensor, if $k - 1$ arguments are set constant and one is free, then that argument obeys the linearity axioms. We can also call

We also have two different types of tensors: namely *symmetric* and *alternating* tensors.

Definition 3.4. A symmetric k-tensor f follows the property

$$f(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}, \ldots, v_{\sigma(k)}) = f(v_1, v_2, v_3, \ldots, v_k).$$

Definition 3.5. An alternating k-tensor follows the property

$$f(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}, \ldots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma) f(v_1, v_2, v_3, \ldots, v_k).$$

The set of all alternating k-tensors is denoted $A_k(V)$.

An alternating k-tensor encodes orientation through $\operatorname{sgn}\sigma$ of volumes. This will be motivated in later sections.

3.2. **The Tensor Product and the Wedge Product.** We can define the symmetrical and alternating properties of k-tensors as operators which act on k-tensors.

We say S to be the operator that 'operates' on a k-tensor f to make it symmetric, and A to make it alternating.

$$Sf = \sum_{\sigma^1,\sigma^2,\dots,\sigma^k} \sigma f, \quad Af = \sum_{\sigma^1,\sigma^2,\dots,\sigma^n} (\text{sgn}(\sigma))\sigma(f)$$

Its proof is quite easy. We must show that $\tau(Zf) = Zf, \quad A, S = Z$, but we will not include the proof here.

Lemma 3.6. *If f is an alternating k-tensor on a vector space V, then $Af = k!f$*

Proof

$$Af = \sum_{\sigma\in S_k} (\text{sgn}(\sigma))\sigma f = \sum_{\sigma\in S_k} (\text{sgn}(\sigma))(\text{sgn}(\sigma))f = \sum_{\sigma\in S_k} f = k!f, \quad |S_k| = k!.$$

Using symmetrising and alternating operators makes defining the wedge product convenient.

Definition 3.7. If f, g are k- and l-tensors respectively, then the tensor product between them creates a new $(k+l)$-tensor

$$(f \otimes g)(v_1, v_2, \dots, v_{k+l}) = f(v_1, v_2, \dots, v_k)g(v_{k+1}, v_{k+2}, \dots, v_{k+l}).$$

The tensor product is associative.

However, if we take two k- and l- *alternating* tensors, the tensor product does not preserve the alternating operator. From this, we motivate the wedge product.

Definition 3.8. For f, g are k- and l-tensors, the wedge product $f \wedge g$ is defined by

$$(f \wedge g)(v_1, v_2, \dots, v_{k+l}) = \frac{1}{k!l!}A(f \otimes g).$$

Notice that

$$\sum_{\sigma\in S_k} (\text{sgn}(\sigma))\sigma f = \sum_{\sigma\in S_k} (\text{sgn}(\sigma))(\text{sgn}(\sigma))f = \sum_{\sigma\in S_k} f = k!f.$$

This applies to two functions, where we obtain

$$\sum_{\sigma\in S_k} k!l!fg.$$

We can avoid factorials by instead rewriting the σs as (k, l)-shuffles, with the following conditions

Conditions 3.9.

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k), \tag{3.1}$$

$$\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(k+l). \tag{3.2}$$

Thus, we can write the wedge product as

$$(f \wedge g)(v_1, \dots, v_{k+l}) = \sum_{\sigma=(k,l)\text{-shuffles } \in S_k} (\text{sgn}(\sigma))(\sigma f)(\sigma g).$$

In this paper, we prefer the first expression for the wedge product.

The wedge product also has two properties, namely *anticommutativity* and *associativity.* We will state their expressions but omit their proofs.

3.2.1. *Anticommutativity.* Anticommutativity states that the wedge product of a k- and l-tensor f and g respectively

$$f \wedge g = \text{sgn}(\tau)(g \wedge f) = (-1)^{kl}(g \wedge f).$$

This is a comment on how the wedge product encodes orientation.

3.2.2. *Associativity.* This property does not need deliberation beyond the following expression.

$$f \wedge (g \wedge h) = (f \wedge g) \wedge h.$$

4. Differential Forms

To understand the motivation of differential forms, we must first understand the issue of integrating over *manifolds.* To do so, we must construct the idea of a manifold.

4.1. **Manifold Theory.** [5] [2]

Note that we will not be proving propositions and lemmas since we focus mainly on efficiently and quickly constructing de Rham cohomology by motivating its components

Definition 4.1. A topological manifold is a $\dim n$ topological space that is Hausdorff and second-countable. We also say that it is locally Euclidean.

Definition 4.2. A topological space is locally Euclidean if there exists a homeomorphism to $\mathbb{R}^n$ of a pair

$$(U, \phi : U \to \mathbb{R}^n)$$

known as a chart. Here, U is the coordinate neighbourhood, and ϕ is the homeomorphism (coordinate map).

We say (U, ϕ) is centered at p if $\phi(p) = 0$. This means that it is in $\ker((U, \phi))$.

Now that we have define a topological manifold, we would like to construct the idea of a smooth manifold. To do so, compatibility of charts needs to be constructed.

Suppose we have two pairs

$$(U, \phi : U \to \mathbb{R}^n),\ (V, \psi : V \to \mathbb{R}^n)$$

where both U, V are open subsets of $\mathbb{R}^n$. Trivially, $U \cap V \in U$, thus it is open, hence $\phi(U \cap V)$ is also an open subset $\in \mathbb{R}^n$. The same argument applies for $\psi(U \cap V)$.

Definition 4.3. Two charts are C^∞-compatible if the maps

$$\phi \cdot \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V),\ \psi \cdot \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

known as transition functions, are both C^∞

Lemma 4.4. *If two charts are compatible with the same atlas, they are compatible with each other.*

Recall compatibility. Compatibility on a smooth manifold ensures that a differential form is defined globally over M if it is defined on one of the charts in a maximal atlas. This allows differential forms to transform appropriately.

Definition 4.5. A C^∞-atlas is the set of all charts $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise C^∞-compatible charts that cover M, in the form of

$$M = \bigcup_\alpha U_\alpha.$$

Definition 4.6. A smooth manifold M is a topological manifold with a maximal atlas, also known as a differentiable structure on M.

Proposition 4.7. *Any atlas on a Euclidean space is always contained in a unique maximal atlas.*

We can define coordinates on both $\mathbb{R}^n$ and a manifold M. The standard coordinates (coordinates represented by standard basis vectors) on $\mathbb{R}^n$ are $r_1, r_2, \ldots, r_n$. The *local coordinates* on U for (U, ϕ) is a chart on a manifold M, are the elements of the range of the homeomorphism from $U \to \mathbb{R}^n$. $\{x_1, x_2, \ldots, x_n\}$ denotes the local coordinates on U. They are functions such that $\{x_1(p), x_2(p), \ldots, x_n(p)\} = \{r_1, r_2, \ldots, r_n\}, \quad p \in U$.

We can now transition to transmuting constructions we did for Euclidean space to manifolds.

Definition 4.8. A function $f : M \to \mathbb{R}$ is C^∞ if $f \circ \phi^{-1}$ a map from $\mathbb{R}^n \to U \subset M \to \mathbb{R} \cong \mathbb{R}^n \to \mathbb{R}$ is C^∞. It should be defined over $\phi(p)$.

f is independent of charts because of smooth transition maps.

Definition 4.9. Let N, M be C^∞ manifolds. Thus, a map $F : N \to M$ is smooth if the map $\psi \circ F \circ \phi^{-1} : \mathbb{R}^n \to U_N \subset N \to U_M \subset M \to \mathbb{R}^m$ for the charts (V, ψ) defined over $F(p) \in M$ and (U, ϕ) defined over $p \in N$ is C^∞ at $\phi(p) \to 0 \in \mathbb{R}^n$.

A key feature of this is that we compare U_N, U_M in terms of $(F^{-1}(V)) \cap U$, and take the homeomorphism ϕ as the starting point of $\mathbb{R}^n \to \mathbb{R}^m$, because the composition is defined only on the intersection of the open sets over N and M. Note that to prove C^∞ of F, we first assume it is continuous so the map from the open set in M to N of the chart is open in N.

Definition 4.10. A diffeomorphism $F : N \to M$ is a bijective smooth map where both F and F^{-1} are smooth. It is analogous to an isomorphism, but for manifolds.

When looking at a homeomorphism $\phi : U \to \mathbb{R}^n,\ U \subset M$ of a manifold M, we can define it in terms of a diffeomorphism F. It is sufficient to check that it is C^∞, as well as the inverse condition.

We can also define smoothness component-wise.

Proposition 4.11. *A map $F : N \to M$ is smooth if all its components $F^1, F^2, \ldots, F^n$ are smooth.*

This is important for understanding component-wise integration on charts, which is the motivation for differential forms later in the paper.

Definition 4.12. Partial derivatives make sense, in terms of *directional derivatives.* We define partial derivatives of $f \in C^\infty$ with respect to x^i as

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) = \frac{\partial f}{\partial x^i}(p) = \frac{\partial (f \circ \phi^{-1})}{\partial r^i}(\phi(p)) = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} (f \circ \phi^{-1}).$$

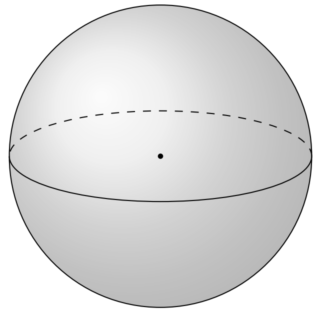

Figure 1. S^2 divided into upper- and low hemispheres

If we consider $r^i : \mathbb{R}^n \to \mathbb{R}$ then we know this maps a vector to its magnitude, which, when evaluated at the homeomorphism over p, returns the component of the standard local coordinates over the open set $U \subset M$ evaluated over p. This means that partial derivatives of smooth functions are analogous to directional derivatives.

Now, we may introduce the Jacobian.

Definition 4.13. Let $F : N \to M$ be a smooth map and $(U, \phi), (V, \psi)$ be charts on N and M respectively, and $F(U) \subset V$. Denote the i^{th} component of F in the chart (V, ψ) by $F^i \in \mathbb{R}$ given by

$$F^i = x^i \circ F = r^i \circ \psi \circ F : U \to \mathbb{R}$$

Then the Jacobian Matrix given by $\left[\frac{\partial F^i}{\partial x^k}\right]$ is the matrix representation of F, relative to the two charts $(U, \phi), (V, \psi)$.

The determinant of the Jacobian is just the scaling factor for transfer between coordinate systems to perform integration. The partials are meant to represent the nudges of the differential forms spanning the space in directions. Keep this idea in mind for when we motivate differential forms..

If we consider the smooth map $F : N \to N$, for N is a $\dim n$ smooth manifold, we can define the Jacobian in terms of the diffeomorphism between two charts of the same manifold. This is the grounding we will use the motivate differential forms in terms of integrating over local coordinate charts.

4.2. **Motivating Differential Forms.** [1] Suppose we have a compact $\dim n$ manifold, and we have a smooth function $f : M \to \mathbb{R}$. Integrating this function can be done over the coordinates of the charts on the manifold.

For the sake of intuition, lets say $M = S^3$.

Take a map from the upper hemisphere $u^+ \to U^+$, an open subset of $\mathbb{R}^2$,

$$\phi : u^+ \to U^+,$$

and a map from the lower hemisphere $u^- \to U^-$, another open subset of $\mathbb{R}^2$,

$$\psi : u^- \to U^-.$$

To integrate f, we can sum the integrals of f over U^+ and U^-,

$$\int_{S^2} f dxdy = \int_{U^+} f \circ \phi^{-1} dxdy + \int_{U^-} f \circ \psi^{-1} dxdy.$$

To check if this is consistent, we pick different coordinates from a different local coordinate chart to check if it is independent of coordinates and hence charts. We can try this for the upper hemisphere by defining a different open subset B^+ and a homeomorphism ϕ_0.

$$\int_{S^2} f dxdy = \int_{B^+} f \circ \phi_0^{-1} dxdy + \int_{U^-} f \circ \psi^{-1} dxdy.$$

We must now check if

$$\int_{U^+} f \circ \phi^{-1} dxdy = \int_{B^+} f \circ \phi_0^{-1} dxdy.$$

We see that

$$\int_{B^+} f \circ \phi_0^{-1} dxdy = \int_{B^+} (f \circ \phi^{-1}) \circ (\phi \circ \phi_0^{-1})\, dxdy,$$

where $\phi \circ \phi_0^{-1}$ is a change of variables map which can be resolved by multiplying the expression with an appropriate determinant of the Jacobian because it is not well-defined. Thus, we must find something to integrate that transfers variables appropriately and consistently. This is our motivation for differential forms.

We may start defining differential forms from differential 1-forms or simply 1-forms. A 1-form is a multilinear function taking one input vector and 'projects' it onto its respective dimensional factor.

Example. In $\mathbb{R}^2$, dx is a differential 1-form $f(v)$, a multilinear function taking a vector v and projecting it onto its x-factor. Similarly, dy is a 1-form that projects vectors onto their y-factors.

The wedge product measures the volume of the parallepiped spanned by vectors inputted into the 1-forms. This is quantised by the determinant of the derivations evaluated with their respective vectors.

Generalising to $\mathbb{R}^n$,

$$(dx^1 \wedge \dots dx^n)(v_1, \dots, v_n) = \det\left[dx^i(v_j).\right]$$

We write differential forms in terms of linear functionals $V \to \mathbb{R}$.

Note that this is simply an alternating tensor. They are suited for volume measurement with the orientation factor.

We show this by some alternating tensor $A : M \to \mathbb{R}$, and a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$T^* A(v_1, \dots, v_k) \to A(T(v_1), \dots, T(v_k)).$$

called the *pullback* of A by T.

We can also write this as

$$\det T A(v_1, \dots, v_k).$$

It is up to the reader now to personalise this information to gain a strong intuition for it. We can now rigorously treat differential forms.

4.3. **Rigorously treating Differential Forms.** [5]

A cotangent space is the vector space of all maps from the tangent space to $\mathbb{R}$

$$\omega_p \in T_p^*(\mathbb{R}^n) = (T_p\mathbb{R}^n)^*,\ \omega_p : T_p(\mathbb{R}^n) \to \mathbb{R}.$$

We can then to define a *covector field* of all ω_p on p. For this, we will need it over some space, say $\mathbb{R}^n$ for simplicity, which must be the union of all cotangent spaces (we can define it over other spaces, especially *manifolds*, but that will be introduced later). Thus, a covector field or tensor field ω is a function that assigns each point in an open subset $U \subset \mathbb{R}^n$ a covector or tensor $\omega_p \in T_p^*(\mathbb{R}^n)$.

$$\omega : U \to \bigcup_{p\in\mathbb{R}^n} T_p^*(\mathbb{R}^n),$$

which is just a map from

$$p \to \omega_p.$$

From now, we will refer to covector fields as tensor fields. This tensor field is a *differential 1-form*, since it consists of 1-tensors. Trivially, all the cotangent spaces are disjoint, since they are defined on different points.

Example. Take $\mathbb{R}^2$. If we define a differential form dx, it tells us

4.3.1. *Differentials.* We can define a 1-form on a C^∞ function called the *differential* of f.

$$(df)_p(X_p) = X_p f$$

A differential applies

$$X_p = \sum_{i=0}^{n} a^i \frac{\partial}{\partial x^i}\bigg|_p$$

to f to give its directional derivative at p in the direction of X_p. So, we get a map between $T_p(\mathbb{R}^n)$ and $C_p^\infty(\mathbb{R}^n)$ to $\mathbb{R}$, formally written as

$$(d)_p : T_p(\mathbb{R}^n) \times C_p^\infty(\mathbb{R}^n) \to \mathbb{R}, \quad f_i \in C_p^\infty(\mathbb{R}^n),\ X_p \in T_p(\mathbb{R}^n).$$

Thus, differentials are really good at spitting out directional derivatives. It is a derivation! We can use differentials *of* derivations to define integration over an open subset $U \subset \mathbb{R}^n$.

4.4. **Differential k-forms.**

Definition 4.14. A differential k-form ω is a function mapping to each point p an element of $A_k(T_p(\mathbb{R}^n))$.

By the original proposition that the wedge products of k-tensors is the basis of the alternating tensor space, similarly the wedge products of these k-tensors is the wedge product of this alternating tensor space. In this case, the basis is the wedge product of differentials

$$dx_p^I = dx_p^{i_1} \wedge dx_p^{i_2} \wedge \cdots \wedge dx_p^{i_k}.$$

Therefore, we may write a k-form acting on a point p as a linear combination from the previous remark,

$$\omega_p = \sum_I a_I(p) dx_p^I,$$

bridging it to the open subset,

$$\omega = \sum_I a_I dx^I.$$

The wedge product of a k- and l-form are defined point-wise.

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p.$$

Thus, we can also write the wedge product in terms of differentials

$$(\omega \wedge \tau)_p = \sum_{I,J} (a_I(p) b_J(p)) dx_p^I \wedge dx_p^J.$$

and so, for the general forms

$$\omega \wedge \tau = \sum_{I,J} (a_I b_J) dx^I \wedge dx^J.$$

Therefore, we now know, by the wedge product, the exterior derivative must map to a $(k+l)$-form

$$\wedge : \Omega^k(\mathbb{R}^n) \times \Omega^l(\mathbb{R}^n) \to \Omega^{k+l}(\mathbb{R}^n).$$

This is a very important result for the *exterior derivative* and the *de Rham complex.*

Remark 4.15. We also see a consequence on the basis of $T_p^*(\mathbb{R}^n)$. Consequently, since

$$\left\{ \frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p, \dots, \frac{\partial}{\partial x^n}|_p \right\}$$

is the basis of $T_p(\mathbb{R}^n)$, the set

$$\left\{ dx^1|_p,\ dx^2|_p, \dots, dx^n|_p \right\}$$

is the basis of $T_p^*(\mathbb{R}^n)$.

Remark 4.16. This section is for intuition only, requiring a rudimentary understanding of representation theory. We can use representation theory on S_k to explain how differential forms encode oriented volume.

Take $G = S^k$, the group of all k-permutations. Define V from (π, V) as $V = \mathbb{C}^k$. We can thus define π_1 as permutations on $I \in GL(V)$.

Define $\pi_2 = \det \pi_1$. This encodes sgn σ, for $\sigma \in S_k$. We can rewrite the alternating k-tensor property as

$$f(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}, \dots, v_{\sigma(k)}) = \pi_2 f(v_1, v_2, v_3, \dots, v_k).$$

In short, we can write it has $\pi_2 f$. This means we are multiplying $\det \pi_1 \in \mathbb{R}$ with the multilinear map $f : V^k \to \mathbb{R}$.

4.5. **The Exterior Derivative.** The exterior derivative is a map

$$d_e : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n).$$

If we have a smooth k-form ω, its differential is a smooth $k+1$-form $d\omega$. Thus, the exterior derivative defines k-differentials in terms of k-tensors!

We may prove that for a k-form ω, its differential $d\omega \in \Omega^{k+1}(\mathbb{R}^n)$. We know that any k-tensor can be written as a linear combination of 1-forms

$$\omega = \sum_I a_I dx^I.$$

Thus, we may write a differential, defined as

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \left(\sum_j \frac{\partial a_I}{\partial x^j} dx^j \right) \wedge dx^I \in \Omega^{k+1}(U).$$

4.6. **Closed & Exact Forms.**

Definition 4.17 (Closed Forms)**.** A closed form is a k-form such that its differential $d\omega = 0$.

Essentially, a differential describes how a k-form changes throughout U. If a k-form is closed, it is constant throughout U.

Definition 4.18 (Exact Forms)**.** An exact form is a k-form ω such that $\exists \tau$, a $k-1$-form such that $\omega = d\tau$.

We note a very important condition between closed and exact forms. Every exact form is closed, because $d\tau$ can just be a 0-form, thus τ is also a 0-form, but the converse may not be true. This is crucial in defining de Rham cohomology.

5. Chain Complexes, Sequences & the de Rham Complex

[4] A cochain complex $\mathcal{C}$ is a collection of vector spaces $\{C^k\}_{k\in\mathbb{Z}}$ together with a sequence of linear maps $d_. C^k \to C^{k+1}$,

$$\dots \xrightarrow{d_{-2}} C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \dots$$

such that,

$$d_k \circ d_{k-1} = 0, \ \forall k.$$

We call the collection of linear maps the *differential* of the cochain complex.

The vector space $\Omega^*(M)$ of differential forms on a manifold M with the exterior derivative d is a cochain complex called the de Rham complex,

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots, \quad d \circ d = 0, \quad \text{im}\, d \subset \ker d.$$

The property $d \circ d$ seems obscure in terms of its motivation. One argument is that, if we map d_{k-1} to the next vector space, we receive the image of it. Now, if we apply d_k to that image, then we're effectively applying d_k to the kernel of d_k, which is 0, since $\text{im}\, d_{k-1} \subset \ker d_k$. However, this creates a circular argument with im $(d_{k-1}) \subset \ker d_k$, with both proving each other (they are equivalent statements). This is just a part of the machinery of cochain complexes.

5.1. **Exact Sequences.** A sequence of homomorphisms on vector spaces

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if $\mathrm{Im}\ f = \ker g$. The logic behind this is that by exact forms $\omega = d\tau$, and they are closed meaning $d\omega = 0$. This means $d\tau \to \omega \to d\omega$ is analogous to this sequence. The rest is self-explanatory. This can transmute into large sequences, where we cannot consider the first and last terms as exact because then it would transmute to infinity.

$$A^0 \xrightarrow{f^0} A^1 \xrightarrow{f^1} A^2 \xrightarrow{f^2} \dots \xrightarrow{f^{n-1}} A^n.$$

Note: A 5-term exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is said to be *short exact*.

If we have an exact sequence of the form

$$0 \xrightarrow{f} B \xrightarrow{g} C$$

where $A = 0$, since it is the zero-vector space, zero vectors are only homomorphic to zero vectors in other space, thus the image $\mathrm{Im}\ f = 0$. We already know the kernel maps to zero, and since zero maps to zero, $\ker g = 0$. Therefore, $\ker g = \mathrm{Im}\ f = 0$, so g is injective.

When $C = 0$,

$$A \xrightarrow{f} B \xrightarrow{g} 0$$

we see that $\ker g = B$ is obvious. However, we also see that $\mathrm{Im}\ f = B$. This means $\mathrm{Im}\ f$ is surjective! (we only say Im f is the same as ker g because we like exact sequences).

Some important propositions which follow from the above two assertions

Proposition 5.1. *For a sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

- f *is surjective* $\iff$ g *is the zero-map*
- g *is injective* $\iff$ f *is the zero-map*

Remark 5.2. The de Rham complex an exact sequence when the space it is defined on has no holes.

Notice that if this is true, then the condition $\mathrm{im}\ d \subset \ker d$ becomes $\mathrm{im}\ d = \ker d$, which motivates $d \circ d = 0$ by $\ker(\mathrm{im}\ d) = 0$. To see how hole detection naturally arises from the *violation* of this property, we must consider Riemann's thoughts on integrating over boundaries and planes.

5.2. **Integrating over boundaries & planes.** Suppose we have a surface with a boundary b_1 around a plane characterised by the hole, as seen in figure 2.

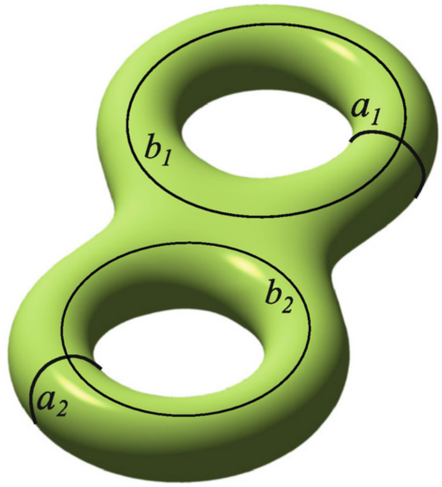

Figure 2. [3] A genus-2 orientable surface with boundaries labeled b_1 and b_2 about the holes.

We can define a *closed* differential form about the neighbourhood ω. We will now say that $b_1 = \partial D$, and the plane it encloses is D. Recall the generalised Stokes' Theorem

$$\int_{\partial D} \omega = \int_D d\omega = 0,$$

since $d\omega = 0$. We label $\int_{\partial D} \omega$ as a period. Recall that a closed form is always exact.

Regarding the other side of the argument, suppose we have an exact form τ such that $\tau = d\sigma$, then

$$\int_{\partial D} \tau = \int_{\partial D} d\sigma = \int_{\partial(\partial D)} d\sigma = \int_0 d\sigma = 0,$$

because the boundary of a boundary is 0 This holds due to Stokes' Theorem. Due to a hole, the condition $\partial(\partial D)$ is violated. This is analogous to $d \circ d$ not holding true in the de Rham complex, so it is not exact, meaning there is a hole. Alternatively, we can say that there is some closed form that is not exact. In a good sentence, non-zero periods of closed forms detect holes.

6. de Rham cohomology [5]

Definition 6.1. A de Rham cohomology is the quotient vector space of closed and exact forms

$$H^k(M) = \frac{Z^k(M)}{B^k(M)},$$

where $Z^k(M)$ is the set of all closed forms and $B^k(M)$ is the set of all exact forms. Alternatively, by the containment of Im $d \in \ker d$,

$$H^k(M) = \frac{\ker d}{\text{Im } d}.$$

The idea behind this leads back to closed and exact forms; all exact forms are closed, but there can exist non-0-closed forms ω that are not exact. What this means is that $d\omega = 0$, but $\omega \neq d\tau$, for any $k-1$-form τ. So 'zero-ness' arbitrarily stems from ω, which seems odd. Recall that closed forms which are not exact reveals holes. Thus, by quotienting the set of all closed forms with the set of all exact, forms, we obtain the closed forms which violate that property.

Note that when $H^k(M) \neq 0$, the quotient does not 'cancel out'. Rather it becomes 0, indicating zero differences between the two sets. This corresponds to $\text{im}\, d = \ker d$ in the de Rham complex.

We may also explain the link between the image and kernel definition and the closed-form and exact-form definition. In the de Rham complex, the exterior derivative is the map $d : \Omega^k(M) \to \Omega^{k+1}(M)$. Recall its meaning in $d : \omega \to d\omega$. It is appropriate to assume the de Rham complex is exact, so it contains exact forms. The image-kernel equivalence implies that all closed forms $d\omega = 0$ (by $\ker d$) are exact $\gamma \in \Omega^{k+1}(M),\ d\omega = \gamma$, and all exact forms are closed. When $\text{im}\, d \subset \ker d$, then not all closed forms are exact. The reasoning is similar to the geometric intuition given previously.

We may now move on to further constructing de Rham cohomology.

The properties of a quotient vector space create a relationship between two closed forms

$$\omega' - \omega = d\tau,$$

where $\omega',\ \omega$ are closed forms and $d\tau$ is an exact form. Regarding this difference, we can explain it in terms of cosets.

6.1. **Cosets.**

Definition 6.2. A coset in $W \subset V$ where both are vector spaces (and obviously, W is a subspace) is the set

$$v + W = \{v + w : w \in W\}.$$

Two cosets are equivalent $v+W \equiv v'+W \iff v' = v+w$. In other words, $v'-v = w \in W$.

$$v' \equiv v \iff v' - v \in W \iff v + W = v' + W.$$

Thus any coset satisfying these conditions is an equivalence class.

Any element of $v'^{\cdots'} + W$ is a representative of $v + W$ if it is equivalent on it.

$\frac{V}{W}$ is a quotient vector space is a set of all cosets of $W \subset V$.

Now for differential forms.

If $H^k(U) = \frac{Z^k(U)}{B^k(U)}$, then $\omega' = \omega + W$, if $W \subset V$, then $W = B^k(U)$ and $V = Z^k(U)$.

$$\therefore w' \equiv w \iff w' - w \in B^k(U)\ by\ w' = w + b^k,\ b^k \in B^k(U).$$

Here, w', w are cohomologous.

We care about cohomologous forms because a period of ω, a k-form, then only depends on its cohomology class. We must understand the role of cycles.

6.2. **Singular Homology.**

[1] Suppose some set $v_0, \dots, v_p$ are any $p+1$ points in $\mathbb{R}^n$. They are said to be affinely independent if they are not contained in any $(p-1)$-dimensional affine subspace of $\mathbb{R}^n$.

Definition 6.3. A geometric p-simplex is a subset of $\mathbb{R}^n$

$$\left\{ \sum_{i=0}^{p} t_i v_i : 0 \leq t_i \leq 1, \ \sum_{i=0}^{p} t_i = 1 \right\},$$

over affinely independent points.

Here, p is the dimension of the simplex. We can think of a simplex as the simplest possible $\dim p$ polytope that exists in $\mathbb{R}^n$. It can also be said as the *convex hull* of affinely independent points.

We call the points of $v_0, \dots, v_p$ the vertices of the simplex. Notation wise, the simplex is given by $[v_0, \dots, v_p]$.

Example. A 2-simplex in $\mathbb{R}^2$ is a triangle. A 0-simplex in $\mathbb{R}^2$ is a point.

Any subset of the affinely independent subset that is a vertex of a simplex is called its face. Vertices of $\dim (p-1)$ are called boundary faces. There are $(p-1)$ boundary faces once we omit every vertex to obtain the boundary faces. We call $\partial_i[v_0, \dots, v_p]$ the face opposite v_i in the original simplex $[v_0, \dots, v_i, \dots, v_p]$.

Definition 6.4. The standard p-simplex

$$\Delta_p = [e_0, \dots, e_p]$$

where $e_0, \dots e_p$ is some subset of the basis standard basis of $\mathbb{R}^n$.

Definition 6.5. A singular p-simplex of a topological space M is a map

$$\sigma : \Delta_p \to M.$$

Definition 6.6. The set of all free abelian groups generated by singular p-simplices $C_p(M)$ is called the singular chain group of M. It has degree p. Elements of this group are called chains singular p-chains, such that

$$c = \sum_i n_i \sigma_i.$$

One can skew the idea of singular p-chains a bit for intuitive purposes, to think of it as creating a 'cover' of singular p-simplices over the structure space, rather than the entire space. For example, we can use singular p-chains as a method to explore the differentiable structure of a smooth manifold. However, this is not a very accurate perspective. It is only meant for intuitive purposes.

Definition 6.7. An affine singular simplex is the restriction of a unique affine map $\mathbb{R}^p \to \mathbb{R}^m$ to Δ_p is given by

$$A(w_0, \dots, w_p),$$

where $\{w_0, \dots, w_p\}$ is a subset of the standard basis of $\mathbb{R}^m$.

This is the result of a potential affinely dependent set in a convex subset $M \subset \mathbb{R}^m$ (since subsets of M need not be affinely independent).

Essentially, this refers to simplices in non-coordinate environments, so we can generalise them to topological spaces using affine simplices.

Definition 6.8. The i^{th} face map is the affine singular simplex of degree $(p-1)$, mapping consecutive-dimensional simplices

$$F_{i,p} : \Delta_{p-1} \to \Delta_p.$$

We say it maps $\Delta_{p-1} \to \partial_i \Delta_p$. Thus, it maps to the face opposite e_i.

Think of it as mapping a singular simplex onto the boundary of an incremental-degree simplex. Intuition-wise, a triangle can be the boundary face of a pyramid (this simplifies the idea of singular simplices to a great extent; a more accurate interpretation would be deforming triangles onto tetrahedra).

The boundary of a singular p-simplex is the singular $(p-1)$-chain

$$\partial\sigma = \sum_{i=0}^{p} (-1)^i \sigma \circ F_{i,p}.$$

Notice the parallel between degrees of simplices and the idea of dimensions of boundaries and planes. We may link the idea of singular chains to the boundary argument $\partial(\partial D) = 0$. In essence, the boundary as an operator ∂ maps a plane to its boundary.

$$\partial : C_p(M) \to C_{p-1}(M).$$

Lemma 6.9. *If c is any singular chain, then $\partial(\partial c) = 0$.*

Proof. Let $\sigma :\to \Delta^{p+1} \to X$ be a singular $(p+1)$-simplex.

$$\partial_{p+1}\sigma = \sum_{i=0}^{p+1} (-1)^i \sigma \circ F_{i,(p+1)},$$

by the previous definition. Then, applying ∂_p, we get

$$\partial_p(\partial_{p+1}\sigma) = \sum_{i=0}^{p+1} (-1)^i \partial_p(\sigma \circ F_{i,(p+1)}) = \sum_{i=0}^{p+1} (-1)^i \sum_{j=0}^{p} (-1)^j \sigma \circ F_{i,(p+1)} \circ F_{j,p}.$$

We can therefore see that, from the identity

$$F_{i,p} \circ F_{j,p-1} = F_{j,p} \circ F_{(i-1),(p-1)}, \quad i > j,$$

which is verifiable by examining the effect of the composition on Δ_{p-2}, each pair (i,j) appears twice with opposing signs, which maps to 0.

We can rewrite the sum as

$$\partial^2(\sigma) = \sum_{i=0}^{p+1} \sum_{j=0}^{p} (-1)^{i+j} \sigma \circ F_{i,(p+1)} \circ F_{j,p}.$$

The term

$$(-1)^{i+j} \sigma \circ F_{i,(p+1)} \circ F_{j,p} = (-1)^{j+(i-1)} \sigma \circ F_{j,(p+1)} \circ F_{(i-1),p}.$$

In the sum, we therefore get

$$(-1)^{i+j} + (-1)^{j+(i-1)} = (-1)^{i+j} + (-1)^{i+j-1} = (-1)^{i+j} - (-1)^{i+j} = 0.$$

∎

A singular p-chain c is called a cycle if $\partial c = 0$, and a boundary of $c = \partial b$ for some singular $(p+1)$-cycle (notice the parallel between cycles and closed forms, this is important to us). We can think of singular p-chains as 'compact submanifolds with no boundary'.

We say this is the homeomorphisms linking abelian groups in the sequence known as the chain complex

$$\cdots \to C_{p+1}(M) \xrightarrow{\partial} C_p(M) \xrightarrow{\partial} C_{p-1}(M) \to \ldots$$

We define the de Rham complex as incremental degrees rather than decremental degrees to signify *differential forms* analogous to singular p-chains. Thus, we see something familiar

Definition 6.10. The p^{th} singular homology group is defined as the quotient vector space of the set of cycles and boundaries

$$H_p(M) = \frac{Z_p(M)}{B_p(M)},$$

where $Z_p(M)$ is the set of all p-cycles and $B_p(M)$ is the set of all p-boundaries.

This is exactly the definition of a de Rham cohomology group but with cycles and boundaries instead. It shows the link between the geometric detection of holes using integrals of differential forms.

We call closed forms *cocycles* and exact forms *coboundaries*. We may now explore the idea of cohomologous differential forms on integration and *de Rham's theorem*.

For any cycle γ,

$$\int_\gamma \omega' - \omega = \int_\gamma d\tau$$

depends only on the cohomology class, since $\partial c = 0$ for a cycle. We cannot use the geometric argument from above over here, thus we can check for holes by integration over cycles via the cohomology class. Thus, we have freed ourselves from the boundary-dependent argument.

Finally, we would like to culminate in a central result of the prolific de Rham's theorem.

Theorem 6.11. *The maximal number of independent cycles equals the maximal number of independent closed forms.*

Essentially, this shows that some $\partial c = d\omega$. It creates links between *singular homology* and *de Rham cohomology.*

7. Conclusion

de Rham cohomology is usually introduced in an abstract manner in different styles which often do not do justice to how it truly exists. A student of de Rham cohomology benefits best through a quick and intuitive foundation discussing all the links and motivations of the subject. However, this paper has only discussed the initial constructions and implications of de Rham cohomology. There exists a vast amount of interesting mathematics centered around this subfield, which unravels its structure beautifully. We suggest the curious reader read about de Rham cohomology, and algebraic topology in general, from John Lee's 'Introduction to Smooth Manifolds' and Loring W. Tu's 'Introduction to Manifolds' as thorough introductions composing the essence of this paper in different perspectives. The reader would best benefit by working with them simultaneously.

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