The Erdős–Kac Theorem

Vikram Sarkar

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if there exists some constant c for which $g(n) \leq cf(n)$ for all $n \in \mathbb{N}$.

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We say $g(n) \ll f(n)$ if $g(n) = O(f(n))$. For two fixed constants a and b, we say $a \ll b$ instead of $a < b$ to indicate that a is "sufficiently less than b."

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Question

How big is $\omega(n)$? That is, roughly speaking, how large do we expect it to be given, say, n is some fixed order of magnitude? (More rigorously speaking, what is an "average order" of $\omega(n)$?)

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Definition

This won't be very central to the main topic, but just to make this notion of "largeness" concrete: f and g have the same **average order** if

$$\lim_{x\to\infty} \frac{\sum_{n\le x} f(n)}{\sum_{n\le x} g(n)} = 1.$$

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Answer to the question

The answer is that $\omega(n) \approx \log\log n$. Meaning, they have the same average order.

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- For each prime p, let $\chi_p = \begin{cases} 1 & p \mid n \\ 0 & p \nmid n \end{cases}$.
- Then, $\omega(n) = \sum_{p \leq n} \chi_p$ since $\chi_p = 1$ iff $p \mid n$.

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- So, we can "approximate" χ_p as $\frac{1}{p}$. Then, $\omega(n) \approx \sum_{p \le n} \frac{1}{p}$.

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Theorem (Mertens, 1874)

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We can use this logic to prove that $\log\log n$ and $\omega(n)$ have the same average order. (Exercise for the reader.)

Further Estimates of $\omega(n)$

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Theorem (Hardy and Ramanujan, 1917)

The **normal order** of $\omega(n)$ is $\log\log n$, that is, for all $\varepsilon > 0$, that is, the density of all $n \in \mathbb{N}$ for which

$$\left|\frac{\omega(n)}{\log\log n} - 1\right| < \varepsilon$$

is 1.

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- Okay, but what about the error term? Namely, what about $\omega(n) - \log\log n$?

Theorem (Turán, 1934)

$$\sum_{n \le x} (\omega(n) - \log\log n)^2 = (1 + o(1))x \log\log x.$$

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Theorem (Erdős, Kac, 1959)

If $a < b$ are in $\mathbb{R} \cup \{-\infty, \infty\}$, the density of the set

$$\left\{ n \in \mathbb{N} \;\middle|\; a \leq \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} \leq b \right\}$$

is

$$\frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} \, \mathrm{d}x.$$

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Okay, what?

Small Simplification

It is not hard to show that it suffices to show the density of the set

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instead. Since $\log\log N = \sum_{p \leq N} p^{-1} + O(1)$, it is also not hard to show that if $A(N) = \sum_{p \leq N} p^{-1}$, it suffices to show that the density of the set

$$\left\{n \in \mathbb{N} \;\middle|\; a \leq \frac{\omega(n) - A(N)}{A(N)^{1/2}} \leq b\right\}.$$

is $\frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} \mathrm{d}x$ as well.

Probability Theory

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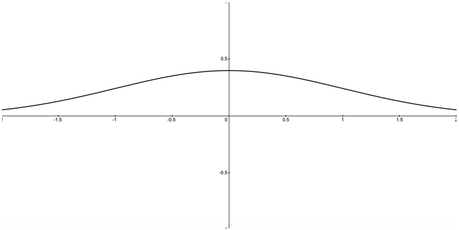

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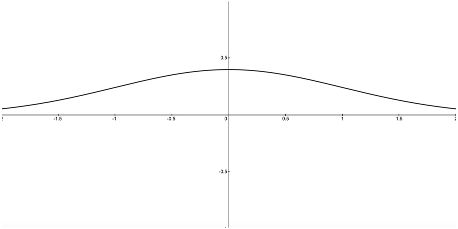

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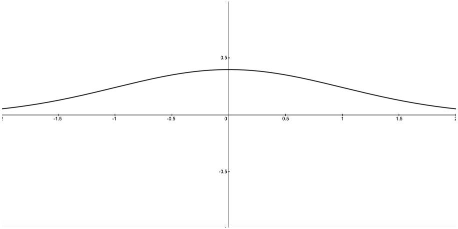

- Looks like a "squished" bell curve.
- Note that $\int_{-\infty}^{\infty} \varphi(x)\ \mathrm{d}x = 1$ due to the infamous $\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}$.

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A **discrete random variable** X has a finite sample space S and fixed probabilities of being each element of S. For example, S could be $\{1, 2, 3\}$, and $\text{Prob}(X = 1) = \frac{1}{3}$, $\text{Prob}(X = 2) = \frac{1}{6}$, $\text{Prob}(X = 3) = \frac{1}{2}$.

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A **continuous random variable** X which has an infinite sample space $S \subseteq \mathbb{R}$ that is equipped with a **probability density function** f has

$$\text{Prob}(a \leq X \leq b) = \int_a^b f(x)\mathrm{d}x.$$

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In our situation, we have the sequence of discrete random variables $\{X_N\}$, for which X_N has sample space $\left\{\frac{\omega(n)-A(N)}{A(N)^{1/2}} \mid n \leq N\right\}$, and each $n \leq N$ has equal probability to be chosen.

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$$\mathbb{E}[X] := \frac{1}{|S|} \sum_{s \in S} s \cdot \mathrm{Prob}(X = s).$$

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$$\mathbb{E}[X^k] := \frac{1}{|S|^k} \sum_{s \in S} s^k \cdot \mathrm{Prob}(X = s).$$

Moments (Continued)

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Theorem (Well-known)

The standard normal distribution is completely determined by all its moments. That is, if there exists some random variable X with

$$\mathbb{E}[X^k] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^k e^{-x^2/2} \, \mathrm{d}x,$$

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Theorem (Well-known)

Suppose $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of random variables. Let X be a random variable that is completely determined by all its moments. Then, if $\mathbb{E}[X_n^k] \to \mathbb{E}[X^k]$ for all $k \geq 1$, then $\mathrm{Prob}(a \leq X_n \leq b) \to \mathrm{Prob}(a \leq X \leq b)$ for all $a < b \in \mathbb{R} \cup \{-\infty, \infty\}$.

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- Now, let X_N be the random variable with sample space $\left\{\frac{\omega(n)-A(N)}{A(N)^{1/2}} \mid n \leq N\right\}$ for which each element of this set has an equal probability of being chosen (if there are somehow repeats, then the probability of $X_n = r$ would be the # of n for which $\frac{\omega(n)-A(N)}{\sqrt{A(N)}} = r$ divided by N).

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- Let X be a random variable with probability density function φ. Then, if we had

$$\mathbb{E}[X_N^k] \to \mathbb{E}[X^k] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^k e^{-x^2/2} \, \mathrm{d}x,$$

  we would then have
  $\mathrm{Prob}(a \leq X_N \leq b) \to \mathrm{Prob}(a \leq X \leq b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} \, \mathrm{d}x.$ where X is a random variable with probability density function φ.

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- Let X be a random variable with probability density function φ. Then, if we had

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  $\mathrm{Prob}(a \leq X_N \leq b) \to \mathrm{Prob}(a \leq X \leq b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, \mathrm{d}x$. where X is a random variable with probability density function φ.
- However, note that $\lim_{N\to\infty}(a \leq X_N \leq b)$ is also the density of $\{n \in \mathbb{N} \mid \frac{\omega(n)-A(N)}{\sqrt{A(N)}}\}$. So showing that the moments of X_N approach the moments of X suffices.

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$$\mu_k := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^k e^{-x^2/2} = \begin{cases} (k-1)!! & k \text{ even}, k > 2 \\ 0 & k \text{ odd} \\ 1 & k = 0 \end{cases}.$$

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This is not very hard to prove; if k is odd then we are integrating a bounded odd function over a symmetric interval so the integral is 0. If k is even, use integration by parts to get a recurrence between μ_k and μ_{k+2}. So, we want to prove $\mathbb{E}[X_N^k] \to \mu_k$.

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- Note that
$$\mathbb{E}[X_N^k] = \frac{1}{N}\sum_{n\le N}\left(\frac{\omega(n)-A(N)}{A(N)^{1/2}}\right)^k.$$
So if this converges to μ_k, then it equals $\mu_k + o(1)$, where the $o(1)$ is with respect to N, not k.

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So if this converges to μ_k, then it equals $\mu_k + o(1)$, where the $o(1)$ is with respect to N, not k.
- So, we want to show that
$$\frac{1}{N}\sum_{n\le N}\left(\frac{\omega(n)-A(N)}{A(N)^{1/2}}\right)^k = \mu_k + o(1).$$

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Upon rearrangement, we want to show that

$$\sum_{n \le N} (\omega(n) - A(N))^k = N(A(N))^{k/2}\mu_k(1 + o(1)).$$

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Theorem

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The important thing here is that $\sum_{p \le N} L_p(n) = \omega(n) - A(N)$. Therefore, we wish to show that

$$\sum_{n \le N} \left(\sum_{p \le N} L_p(n) \right)^k = N(A(N))^{k/2} \mu_k (1 + o(1)).$$

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Now, for each prime p, let $L(p)$ be a random variable for which $L(p) = -\frac{1}{p}$ with probability $\frac{1}{p}$ and $L(p) = 1 - \frac{1}{p}$ with probability $1 - \frac{1}{p}$.

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Theorem

Suppose $N \geq 1$, and $N_S = N^{1/\log(\sqrt{\log\log N}+3)}$. Then,

$$\sum_{p \leq N} L_p(n) = \sum_{p \leq N_S} L_p(n) + O(\log(\sqrt{\log\log N} + 3)),$$

for all $n \leq N$.

Relating L_p to $L(p)$

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Theorem

Let $\pi(n)$ be the number of primes $\leq n$. Then, for any positive integer k, we have

$$\sum_{n \leq N} \left(\sum_{p \leq N_S} L_p(n) \right)^k = N\mathbb{E}\left[\left(\sum_{p \leq N_S} L(p) \right)^k \right] + O(3^k \pi(N_S)^k)$$

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So we have expressed something very similar to what we want (the LHS) as something relating the random variable $L(p)$. Right now, the right hand side is not very useful since the main expression is stuck in an expected value. So we need a good way to approximate it.

Moment Generating Functions

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Proof

It is clear that M_X is analytic on $\mathbb{C}$. Note that from the Taylor Series expansion of e^x,

$$M_X(z) = \mathbb{E}[e^{zX}] = \mathbb{E}\left[\sum_{k\geq 0} \frac{(zX)^k}{k!}\right] = \sum_{k\geq 0} \mathbb{E}\left[\frac{(zX)^k}{k!}\right] = \sum_{k\geq 0} \frac{\mathbb{E}[X^k]}{k!} z^k.$$

Thus $M_X^{(k)}(0) = \mathbb{E}[X^k]$, as desired.

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- Then,

$$\begin{aligned} M_{B_N}(t) &= \mathbb{E}\left[e^{t\sum_{p \leq N} L(p)}\right] \\ &= \prod_{p \leq N} \mathbb{E}\left[e^{tL(p)}\right]. \end{aligned}$$

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$$= \prod_{p \le N} \mathbb{E}\left[e^{tL(p)}\right].$$

- Now, for $|t| \le \frac{1}{2}$, we can say by Taylor Series that

$$e^{tL(p)} = 1 + tL(p) + \frac{t^2}{2} L(p)^2 + O\left(|t|^3 |L(p)|^3\right),$$

so

$$\mathbb{E}\left[e^{tL(p)}\right] = 1 + \frac{p-1}{2p^2} t^2 + O\left(\frac{|t|^3}{p}\right).$$

Approximating RHS (Continued)

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- One can then take the product over all $p \leq N$ and then bound to obtain that
$$M_{B_N}\left(\frac{t}{\sqrt{\log\log N}}\right) \to e^{t^2/2}$$
uniformly on some disk centered at 0. Then, by complex analysis magic, the derivatives converge uniformly as well.

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uniformly on some disk centered at 0. Then, by complex analysis magic, the derivatives converge uniformly as well.

- Then we can compute

$$\mathbb{E}[B_N^k] = M_{B_N}^{(k)}(0) = (1 + o(1))\mu_k(\log\log N)^{k/2},$$

since the k^{th} derivative of $e^{x^2/2}$ at 0 is μ_k (!!) from the Taylor Series expansion of e^x.

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Theorem

$$\sum_{n\le N}\left(\sum_{p\le N} L_p(n)\right)^k - \sum_{n\le N}\left(\sum_{p\le N_S} L_p(n)\right)^k = o(N(\log\log N)^{k/2}).$$

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Theorem

$$\sum_{n \le N} \left(\sum_{p \le N} L_p(n) \right)^k - \sum_{n \le N} \left(\sum_{p \le N_S} L_p(n) \right)^k = o(N(\log\log N)^{k/2}).$$

Now, we can use the result we obtained above to get the sum for $p \le z$.

The Finish (Continued)

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Note that

$$\begin{aligned}(\log\log N)^{1/2} - (\log\log N_S)^{1/2} &\le (\log\log N - \log\log N_S)^{1/2} \\ &= \left(\log\log N - \log\left(\frac{\log N}{\log(\sqrt{\log\log N}+3)}\right)\right)^{1/2} \\ &= (\log\log(\sqrt{\log\log N}+3))^{1/2}\end{aligned}$$

The Finish (Continued)

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&= (\log\log(\sqrt{\log\log N}+3))^{1/2}
\end{aligned}
$$

so by the Binomial Theorem,

$$
(\log\log N)^{k/2} - (\log\log N_S)^{k/2} = O\left(k(\log\log N_S)^{\frac{k-1}{2}}(\log\log(\sqrt{\log\log N+3}))\right),
$$

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Putting it altogether, we have

$$\begin{aligned}\sum_{n\le N}\Big(\sum_{p\le N} L_p(n)\Big)^k &= (1+o(1))N(\log\log z)^{k/2}\mu_k + O(3^k\pi(N_S)^k)\\ &+ o(N(\log\log N)^{k/2})\\ &= (1+o(1))N(\log\log N)^{k/2}\mu_k + O(k(\log\log z)^{\frac{k-1}{2}}) + \cdots\\ &= (1+o(1))N(\log\log N)^{k/2}\mu_k\\ &= (1+o(1))N(A(N))^{k/2}\mu_k,\end{aligned}$$

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Putting it altogether, we have

$$\begin{aligned}\sum_{n\le N}\Big(\sum_{p\le N} L_p(n)\Big)^k =& (1+o(1))N(\log\log z)^{k/2}\mu_k + O(3^k\pi(N_S)^k)\\ &+o(N(\log\log N)^{k/2})\\ =&(1+o(1))N(\log\log N)^{k/2}\mu_k + O(k(\log\log z)^{\frac{k-1}{2}})+\cdots\\ =&(1+o(1))N(\log\log N)^{k/2}\mu_k\\ =&(1+o(1))N(A(N))^{k/2}\mu_k,\end{aligned}$$

so we are done. ■

Thanks!

Any questions?