On Modular Forms and Hecke Operators

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Modular Forms

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- A modular form of weight k satisfies $f(\gamma\tau) = (c\tau + d)^k f(\tau)$.
- We denote the set of modular forms over Γ as $\mathcal{M}_k(\Gamma)$.

q-expansions

The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$ gives the relation $f(\tau + 1) = f(\tau)$ meaning each modular form is 1-periodic. We can write the Fourier expansion for f as

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n$$

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Examples

Given a complex lattice $\Lambda \subset \mathbb{C}$ generated by $\langle 1, \tau \rangle$ we say that the *Eisenstein series* of weight $k \geq 2$ is

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Another important modular form is the *discriminant function*. This is defined as

$$\Delta(\tau) = (40G_4)^3 - 27(140G_6)^2$$

This is a weight 12 cusp form.

Congruence subgroups

Sometimes we want to study modular forms that satisfy the $(c\tau + d)^k$ relation only for a subgroup $\Gamma \subset SL_2(\mathbb{Z})$ instead of the whole thing. These are called *congruence subgroups*.

Congruence subgroups - continued

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A modular form defined over $\Gamma \supset \Gamma(N)$ is said to have *level* N. Modular forms over $SL_2(\mathbb{Z}) = \Gamma(1)$ have level 1.

Modular curves

A modular curve $Y(\Gamma)$ for congruence subgroup Γ is defined as the orbit space $\Gamma\backslash\mathbb{H} = \{\Gamma\tau : \tau \in \mathbb{H}\}$. These modular curves can be given the structure of a compact Riemann surface when we add the *cusps* to it and these compact Riemann surfaces are denoted as $X(\Gamma)$.

Vector spaces of modular forms

The space of modular forms and cusp forms actually form vector spaces over $\mathbb{C}$. We can use modular curves and the Riemann-Roch theorem to derive a formula for the dimension of these vector spaces.

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Theorem (Dimension Formula)

Let Γ be a congruence subgroup, let g denote the genus of $X(\Gamma)$ and let ε_2, ε_3, and ε_∞ denote the number of "elliptic points" of orders 2 and 3, and the number of cusps, respectively. Then for any even integer $k \geq 2$, the space of modular forms of weight k satisfies

$$\dim \mathcal{M}_k(\Gamma) = (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \varepsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \varepsilon_3 + \frac{k}{2}\varepsilon_\infty,$$

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For example the space of cusp forms of weight 12 has dimension 1.

Hecke Operators

What is a Hecke operator?

A *Hecke operator* is a linear operator from $\mathcal{M}_k \to \mathcal{M}_k$. It acts as a kind of "averaging operator". For simplicity's sake we'll only look at Hecke operators on level 1 modular forms. We define the operators T_p and T_n where p is a prime and n is just a positive integer.

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$$T_p f(\tau) = p^{k-1} f(p\tau) + \frac{1}{p} \sum_{j=0}^{p-1} f\left(\frac{\tau + j}{p}\right).$$

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It can be shown that T_p acts on the Fourier coefficients in the following way:

$$a_n(T_p f) = a_{pn}(f) + p^{k-1} a_{n/p}(f).$$

Hecke operators - continued

Let's define T_{p^r}, where p is a prime, recursively as follows:

$$T_{p^r} = T_p T_{p^{r-1}} - p^{k-1} T_{p^{r-2}}.$$

From here we can extend the definition to T_n where T_1 is the identity and

$$T_n = \prod T_{{p_i}^{e_i}} \text{ where } p_i \text{ are prime and } n = \prod p_i^{e_i}.$$

Notice this means that we have $T_n T_m = T_{nm}$ when $(n, m) = 1$.

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With some simplification we can see that T_n acts on the Fourier coefficients of f as

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From this we get that $a_1(T_n f) = a_n(f)$

Ramanujan tau conjecture

Theorem (Ramanujan tau conjecture)

Given the modular discriminant function

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=0}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

(a) $\tau(n)\tau(m) = \tau(mn)$ *if* $(m, n) = 1$

(b) *For prime* p *and* $j \in \mathbb{N}$ *we have* $\tau(p^{j+1}) = \tau(p)\tau(p^j) - p^{11}\tau(p^{j-1})$

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**This was proved as a result of the Weil conjectures and is beyond the scope of this paper.*

Proof.

We know that the Hecke operator takes $\mathcal{S}_{12}(SL_2(\mathbb{Z})) \to \mathcal{S}_{12}(SL_2(\mathbb{Z}))$ and recall that $\dim S_{12}(SL_2(\mathbb{Z})) = 1$. This means that if we apply T_n to this function, we must have $T_n\Delta = \lambda_n\Delta$. Recall from before that $a_1(T_nf) = a_n$ so we get

$$a_1(T_n\Delta) = a_n(\Delta).$$

This means that

$$T_n\Delta = \tau(n)q + \cdots .$$

We must have that $T_n\Delta = \tau(n)\Delta$. Since $T_mT_n = T_{mn}$ if $(m,n) = 1$ and we know that the *eigenvalue* of T_n is $\tau(n)$ we get that $\tau(m)\tau(n) = \tau(mn)$ when $(m,n) = 1$.

Ramanujan tau conjecture - continued

Let's apply T_p to Δ. If we look at the coefficient of p^r(which is $\tau(p)\tau(p^r)$ because $\tau(p)$ is the eigenvalue) we get

$$\tau(p)\tau(p^r) = a_{p^{r+1}}(\Delta) + p^{11}a_{p^{r-1}}(\Delta)$$

Rearranging, we get the formula

$$\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1}).$$

□

Conclusion

Thank you for listening!