ON MODULAR FORMS AND HECKE OPERATORS

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Abstract. This paper provides an expository account of Hecke operators in the theory of modular forms. We begin by introducing modular forms and congruence subgroups, and describe how modular forms can be viewed as functions on modular curves. Hecke operators are then defined via double cosets and studied through their action on Fourier expansions, inner products, and eigenforms. Finally, we explore their geometric interpretation as correspondences between modular curves.

Contents

1. Introduction

The study of modular forms sits at the crossroads of analysis, algebra, and geometry. These holomorphic functions on the upper half-plane, symmetric under the action of certain groups of matrices, and have connections across number theory. A central theme in their theory is the action of Hecke operators, linear operators that act on spaces of modular forms. Hecke operators and modular forms were pivotal to the proof of the Taniyama-Shimura conjecture which implied Fermat's Last Theorem.

A major historical motivation for understanding Hecke operators comes from Ramanujan's Δ-function, defined by the q-expansion

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n, \quad \text{where } q = e^{2\pi i\tau}.$$

Ramanujan conjectured a number of remarkable properties of the Fourier coefficients $\tau(n)$, including multiplicativity and bounds of the form $|\tau(p)| \leq 2p^{11/2}$. The first two parts were proved by Mordell in 1917 using Hecke [Mor17], while the final bound was proven decades later by Deligne in 1974 as a consequence of his proof of the Weil conjectures [Del74].

In this paper, we explore the theory of Hecke operators on modular forms from both algebraic and geometric perspectives. We begin in Section 2 by recalling the definitions of modular forms and congruence subgroups, and interpreting modular forms as sections of line bundles over modular curves. In Section 3, we derive the dimension formula for spaces of modular forms and cusp forms, using the Riemann-Roch theorem on modular curves.

In Section 4, we introduce Hecke operators via their double coset construction and study their action on q-expansions. We define the operators T_n and $\langle d \rangle$. Section 5 discusses the Petersson inner product and the spectral theory of Hecke operators, ending with the diagonalizability of the Hecke algebra on spaces of cusp forms. We also define the idea of a newform.

Finally, in Section 6, we give a geometric interpretation of Hecke operators in terms of correspondences between modular curves. We define modular curves $X_0(N)$ as moduli spaces of elliptic curves with level structure and construct the Hecke correspondence T_p via maps between $X_0(N)$ and $X_0(pN)$. We conclude with a discussion of the Eichler–Shimura congruence relation, which expresses the Hecke operator T_p as the sum of Frobenius and its transpose on the mod p reduction of the Jacobian of a modular curve.

2. Modular Forms, Congruence Subgroups, and Modular Curves

Before looking at the definition of a modular form, it is beneficial to take a look at the study of *elliptic functions*, or doubly-periodic functions. These are functions f on $\mathbb{C}$ that satisfy

$$f(z+\omega_1) = f(z),$$
$$f(z+\omega_2) = f(z)$$

where ω_1, ω_2 are the two periods of the function, and their ratio is nonreal. These two periods generate a lattice.

Definition 2.1. *We define a* ***complex lattice*** Λ *as* $\Lambda = \{n\omega_1 + m\omega_2 : m, n \in \mathbb{Z}\}$ *where* $\omega_1/\omega_2 \notin \mathbb{R}$ *and* ω_1, ω_2 *are known as the fundamental periods.*

Consider the function $\wp(z)$, the Weierstrass $\wp$-function, which is doubly-periodic with fundamental periods ω_1, ω_2, defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda}{}' \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

(where the primed summation excludes 0). The $-\frac{1}{\omega^2}$ term is added to make the sum converge as it makes the summand approximately $z\omega^{-3}$, which converges over a two dimensional sum.

Let's look at the Laurent expansion of the $\wp$ function and to do that we first need to define something called the *Eisenstein series.*

Definition 2.2. *We define the* ***Eisenstein series*** $G_k(\Lambda)$ *to be*

$$G_k(\Lambda) = \sum_{\omega \in \Lambda}{}' \frac{1}{\omega^k}.$$

Notice that when k is odd, positive and negative terms cancel out causing $G_k = 0$. Since the sum is over a lattice, it only converges when $k > 2$.

One can check that through some simplification we arrive at

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(2k+1)G_{2k+2}z^{2k}.$$

One interesting property that the Weierstrass $\wp$ function satisfies is its differential equation. We can show by matching coefficients of the Laurent expansion of $\wp$ and $\wp'$ that

$$(\wp')^2 = \wp^3 - g_2\wp - g_3$$

where $g_2 = 60G_4$ and $g_3 = 140G_6$.

Given a lattice Λ, we can form the quotient space $\mathbb{C}/\Lambda$, which is a compact Riemann surface of genus 1 (meaning it has one hole). This quotient is known as a complex torus (think about gluing the edges of the fundamental parallelogram into a cylinder and then gluing the ends to form a donut shape). Let Λ be a lattice $\langle\omega_1, \omega_2\rangle$. Then for some nonzero complex number λ, $\mathbb{C}/(\lambda\Lambda)$ is isomorphic to $\mathbb{C}/\Lambda$ meaning that we only care about lattices up to scaling. From now on we'll use $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ where $\tau \in \mathbb{H}$, the complex upper half-plane. From a complex analysis viewpoint, these tori are known as *elliptic curves.* Algebraically an elliptic curve is $y^2 = x^3 + ax + b$. We can see that replacing (x, y) with $(\wp(z), \wp'(z))$ does the trick.

The question now is when are two complex tori $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ isomorphic? We can transform the bases of the lattices with a matrix in $GL_2(\mathbb{Z})$, and it will only work both ways when the matrix is invertible. Since we can replace γ with $-\gamma$, we get the 2×2 matrices with integer entries and deteriminant 1. This means the lattices are homothetic when

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1$$

The matrices of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belong to something called the *special linear group.*

Definition 2.3. *The* ***special linear group*** $SL_2(\mathbb{Z})$ *is the set of matrices* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *such that* $a, b, c, d \in \mathbb{Z}$ *and* $\det(\gamma) = 1$*. The matrix γ acts on a complex number z as* $\gamma z = \frac{az+b}{cz+d}$.

Now let's see what happens when we plug in $\gamma\tau$ to our Eisenstein series. We have

$$G_k\left(\frac{a\tau + b}{c\tau + d}\right) = \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m + n(a\tau + b)/(c\tau + d))^k} = \sum_{m,n\in\mathbb{Z}}{}' \frac{(c\tau + d)^k}{((mc + na)\tau + (nb + md))^k}.$$

Factoring out the $(c\tau + d)^k$ the remaining part is just $G_k(\tau)$ because since $\det(\gamma) = 1$ there is a bijection $(m, n) \to (nb + md, mc + na)$. This means we have

$$G_k(\gamma\tau) = (c\tau + d)^k G_k(\tau).$$

This is the first example of a *modular form*, which is a function that satisfies this kind of relation. Now, we can finally give a formal definition.

Definition 2.4. *A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is **weakly modular** of weight k if*

$$f(\gamma\tau) = (c\tau + d)^k f(\tau)$$

where $\tau \in \mathbb{H}$ and $\gamma \in SL_2(\mathbb{Z})$.

We say that the factor of automorphy $j(\gamma, \tau) = c\tau + d$.

The matrices $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ are generators for $SL_2(\mathbb{Z})$ so for a function to be weakly modular with weight k it suffices to check that

$$f(\tau + 1) = f(\tau) \text{ and } f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau)$$

Definition 2.5. *A function $f : \mathbb{H} \to \mathbb{C}$ is a **modular form** of weight k if it is*

(1) holomorphic on $\mathbb{H}$
(2) weakly modular of weight k
(3) holomorphic at ∞ (known as the cusp*)*

The space of modular forms of weight k is denoted as $\mathcal{M}_k(SL_2(\mathbb{Z}))$.

The space $\mathcal{M}_k(SL_2(\mathbb{Z}))$ forms a vector space over $\mathbb{C}$ and later on we will discuss ways to find the dimension of such vector spaces and how to apply linear operators called Hecke operators.

Since we have that $f(\tau + 1) = f(\tau)$, the function f is periodic with period 1. This means that we can write its Fourier series as

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n$$

where $q = e^{2\pi i \tau}$.

Definition 2.6. *A **cusp form** of weight k is a modular form where $a_0 = 0$ in the Fourier expansion of f. This can also be interpreted as the form vanishing at the cusp, $\lim_{\operatorname{Im}\tau \to \infty} f(\tau) = 0$. The space of cusp forms is denoted $\mathcal{S}_k(SL_2(\mathbb{Z}))$.*

One very important example of a cusp form that will show up later is the modular discriminant function. This is defined as

$$\Delta(\tau) = g_2^3 - 27g_3^2$$

where g_2 and g_3 are the coefficients from the $\wp$ differential equation. Since these are made up of g_2, which is weight 4, and g_3 with weight 6, both terms are weight 12 giving us that Δ is a weight 12 cusp form. Notice that given forms f, g of weight k, l respectively we have $\mathcal{M}_k \times \mathcal{M}_l \to \mathcal{M}_{k+l}$ given by $f, g \mapsto fg$.

Sometimes it isn't only interesting to study modular forms over the entirety of $SL_2(\mathbb{Z})$ but instead just a portion of it. We denote certain subgroups of $SL_2(\mathbb{Z})$ called *congruence subgroups* as Γ.

Definition 2.7. *The **principle congruence subgroup** of level N is defined as*

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{N} \right\}$$

There are two other important congruence subgroups that arise which are Γ_0 and Γ_1 which are defined as follows:

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod N, a \equiv d \equiv 1 \pmod N \right\}$$

Definition 2.8. *More generally, a subgroup $\Gamma \subset SL_2(\mathbb{Z})$ is called a* ***congruence subgroup*** *if for some positive integer N, $\Gamma(N) \subset \Gamma$. We call Γ a congruence subgroup of level N.*

Define the weight k operator $[\gamma]_k$ for $\gamma \in SL_2(\mathbb{Z})$ on a function f to be $(f[\gamma]_k)(\tau) = j(\gamma, \tau)^{-k} f(\gamma\tau)$. Say that a function is *weight-k invariant* with respect to Γ if $f[\gamma]_k = f$ for all $\gamma \in \Gamma$.

Now we can modify our definition of a modular form slightly to take congruence subgroups into account.

Definition 2.9. *Let Γ be a congruence subgroup. A function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k with respect to Γ if*

(1) f is holomorphic
(2) f is weight-k invariant with respect to Γ
(3) $f[\alpha]_k$ is holomorphic at ∞ for all $\alpha \in SL_2(\mathbb{Z})$

If $a_0 = 0$ in the Fourier expansion of $f[\alpha]_k$ for all $\alpha \in SL_2(\mathbb{Z})$ then f is a cusp form of weight k with respect to Γ. The spaces of these forms are denoted $\mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k(\Gamma)$ respectively. If Γ is a congruence subgroup of level N the modular form is said to have level N.

Observe that not all congruence subgroups Γ will contain the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, which takes $\tau \to \tau + 1$, which allows us to write the Fourier expansion with period 1. It is the case though that we will have a matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & h \end{smallmatrix}\right) \in \Gamma$ taking $\tau \to \tau + h$ for some minimal positive integer h. Then the Fourier expansion becomes $f(\tau) = \sum_{n=0}^{\infty} a_n q_h^n$ where $q_h = e^{2\pi i \tau / h}$

Another important concept that will come up later is the idea of a *modular curve.* A modular curve is the quotient space of the orbits of Γ, a congruence subgroup.

Definition 2.10. *Given a congruence subgroup Γ two points $\tau, \tau' \in \mathbb{H}$ are considered Γ-equivalent if they're in the same orbit under the action of Γ. A* ***modular curve*** *$Y(\Gamma)$ is defined as*

$$Y(\Gamma) = \Gamma \backslash \mathbb{H} = \{\Gamma\tau : \tau \in \mathbb{H}\}.$$

We denote $Y(N) = Y(\Gamma(N))$, $Y_0(N) = Y(\Gamma_0(N))$, and $Y_1(N) = Y(\Gamma_1(N))$.

This is actually a *Riemann surface.* The issue is that a lot of nice theorems and results (like the Riemann-Roch theorem used later to find the dimension of $\mathcal{M}_k$) only show up when the Riemann surface is *compact*, which these modular curves currently aren't. Luckily, we can give modular curves the structure of a compact Riemann surfaces by adding finitely many cusp points (think about adding ∞ to $\mathbb{C}$ to get the Riemann sphere $\widehat{\mathbb{C}}$, which is compact). We denote the compact Riemann surface associated with the modular curve $Y(\Gamma)$ as $X(\Gamma)$.

3. The Dimension Formula

We saw before that $\mathcal{M}_k$ and $\mathcal{S}_k$ are both vector spaces. One thing of interest to us when discussing vector spaces is the *dimension* of the space.

In this section we will prove the dimension formula for $\mathcal{M}_k(\Gamma)$ given some congruence subgroup Γ using the *Riemann-Roch Theorem*. To do this we first need to define a few things.

Definition 3.1. *Define a $k/2$-fold differential form as $f(z)(\mathrm{d}z)^{k/2}$ for some function f. The space of these is denoted as $\Omega^{k/2}$*

Let f be a modular form of weight k. We already know that $f(\gamma\tau) = j(\gamma,\tau)^k f(\tau)$ but we also have that

$$\mathrm{d}(\gamma\tau) = \mathrm{d}\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{a(c\tau+d)-c(a\tau+b)}{(c\tau+d)^2}\mathrm{d}\tau = \frac{ad-bc}{(c\tau+d)^2}\mathrm{d}\tau = j(\gamma,\tau)^{-2}\mathrm{d}\tau.$$

This means that if we take $f(\tau)(d\tau)^{k/2}$ and apply γ we will get $j(\gamma,\tau)^k f(\tau) j(\gamma,\tau)^{-k}\mathrm{d}\tau = f(\tau)(\mathrm{d}\tau)^{k/2}$ showing that the for a modular form of weight k, the $k/2$-fold differential form is Γ invariant.

Next we introduce the Riemann-Roch theorem; but first it is important to quote some facts from the theory of Riemann surfaces.

Proposition 3.2. *Let X be a compact Riemann surface. Then the following are true*

(1) Holomorphic functions on X are constant,
(2) Meromorphic functions on X have finitely many poles and zeroes,
(3) Meromorphic functions on X have the same number of poles as zeroes.

Now we can define something called a *divisor*.

Definition 3.3. *A **divisor** D on a compact Riemann surface X is defined as*

$$D = \sum_{x_i \in X} n_i \cdot x_i$$

where $n_i \in \mathbb{Z}$. There are few nonzero n_i.

Note that when we "add" each term we aren't actually adding the mulitples of points but instead the points are acting like basis vectors. When we compare two divisors D_1 and D_2 and say that $D_1 \geq D_2$ we are saying that every coefficient in D_1 is greater than the coefficient for the same point in D_2. This is called a *formal sum*.

The group of divisors $\mathrm{Div}(X)$ is the free abelian group on the points of X.

Definition 3.4. *The **degree** of a divisor D is defined as $\deg(D) = \sum_{x_i \in X} n_i$.*

We say that the order of a function at a point is the order of the pole or zero there. If a function f has an order n zero at x we write $\mathrm{ord}_x(f) = n$. Likewise if f has a pole of order m at x we write $\mathrm{ord}_x(f) = -m$. If f has neither a zero or a pole the order is 0.

Definition 3.5. *The divisor of f, for a meromorphic function on X is defined as*

$$\mathrm{div}(f) = \sum_{x_i \in X} \mathrm{ord}_{x_i}(f) \cdot x_i$$

Note that $\text{ord}_{x_i}(f) \neq 0$ for finitely many x_i and that $\deg(\text{div}(f))$ always equals 0 by Proposition 3.2. Let $\mathbf{C}(X)$ be the set of meromorphic functions on X. For a divisor D define the vector space $L(D)$ to be

$$L(D) = \{f \in \mathbf{C}(X), \text{div}(f) + D \geq 0\} \cup \{0\}.$$

We denote the dimension of this vector space as $\ell(D)$. We also say that a *canonical divisor* is $\text{div}(\omega)$ for a 1-form ω. Recall that the *genus* of a Riemann surface is the number of holes. Now we have the necessary background to state the Riemann-Roch Theorem.

Theorem 3.6 (Riemann-Roch Theorem)**.** *Let X be a compact Riemann surface and let g denote its genus. Then for a divisor D and canonical divisor K we have*

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g$$

From here, it will help us to extract some corollaries of this theorem to make it more applicable to our purpose.

Corollary 3.6.1. *The following statements follow from Riemann-Roch*

(1) $\ell(K) = g$
(2) $\deg(K) = 2g - 2$
(3) If $\deg(D) < 0$ *then* $\ell(D) = 0$
(4) If $\deg(D) > 2g - 2$ *then* $\ell(D) = \deg(D) + 1 - g$

Proof. Taking $D = 0$, we see that $L(D)$ is the space of holormorphic functions on X. Since X is compact we know that $\ell(D) = 1$. Plugging in we get $1 - \ell(K) = 0 + 1 - g \implies \ell(K) = 1$.

Then if we take $D = K$ we get $\ell(K) - \ell(0) = \deg(K) + 1 - g$. Since $\ell(K) = g$ we will have $\deg(K) = 2g - 2$.

Suppose that $\ell(D) > 0$. Then there's some nonzero $f \in L(D)$ so $\text{div}(f) \geq -D$ which shows that $\deg(D) \geq 0$. This proves part (3).

If $\deg(D) > 2g - 2$ then by part (2) we get $\deg(K - D) < 0$ showing that $\ell(K - D) = 0$ by part (3). This proves part (4). □

Generally, for $k/2$-fold differential forms $\deg(\omega) = k(g - 1)$.

The space of autormorphic forms (meromorphic modular forms) of weight k is denoted as $\mathcal{A}_k(\Gamma)$ and it is true that this space is a one dimensional vector space over $\mathbf{C}(X(\Gamma))$ meaning that for $f \in \mathcal{A}_k(\Gamma)$,

$$\mathcal{A}_k(\Gamma) = \mathbf{C}(X(\Gamma))f.$$

Then the subspace $\mathcal{M}_k$ are the holomorphic forms in $\mathcal{A}_k$ so we can say that

$$\begin{aligned}\mathcal{M}_k(\Gamma) &= \{f_0 f \text{ for } f_0 \in \mathbf{C}(X(\Gamma)) \text{ and } f \in \mathcal{A}_k(\Gamma) : \text{div}(f_0 f) \geq 0\} \cup \{0\} \\ &\cong \{f_0 \in C(X(\Gamma)) : \text{div}(f_0) + \text{div}(f) \geq 0\} \cup \{0\}\end{aligned}$$

It looks this is just $L(\text{div}(f))$ allowing us to apply Riemann-Roch and get the dimension of $\mathcal{M}_k$. The issue here is that not all of the coefficients in the divisor are integers. This is because of *elliptic points*. Given the stabilizer group $\Gamma_\tau = \{\gamma \in \Gamma : \gamma\tau = \tau\}$, τ is an elliptic point with period e equal to the order of γ in the quotient group $\Gamma/\{\pm I\}$. Due to the group structure of congruence subgroups we only ever get elliptic points of order 2 or 3. A *cusp* on $X(\Gamma)$ is a Γ-equivalence class of rationals gotten from applying $\gamma \in \Gamma$ to ∞.

We will need the following lemma to help us get the final formula

Lemma 3.1. [DS05] *Let $f \in \mathcal{A}_k(\Gamma)$ and let ω be the corresponding $k/2$-fold differential form on $X(\Gamma)$. Let $\pi : \mathbb{H} \to X(\Gamma)$ then*

(1) If τ is an elliptic point with period e

$$\mathrm{ord}_\tau(f) = \mathrm{ord}_{\pi(\tau)}(\omega)e + \frac{k}{2}(e-1).$$

(2) If τ is a cusp point then

$$\mathrm{ord}_\tau(f) = \mathrm{ord}_{\pi(\tau)}(\omega) + \frac{k}{2}.$$

(3) For all other τ

$$\mathrm{ord}_\tau(f) = \mathrm{ord}_{\pi(\tau)}(\omega).$$

Proof sketch. The third part is clear so we only have to check the first 2. Locally, near an elliptic point, π takes $z \mapsto z^e$. This means that $\omega = h(u)(\mathrm{d}u)^{k/2}$ where $u = z^e$. Then $\omega = h(z^e) \cdot e^{k/2} z^{(e-1)(k/2)} (\mathrm{d}z)^{k/2}$ so $f(z) = e^{k/2} z^{(e-1)(k/2)} h(z^e)$. We will get $\mathrm{ord}_z(f) = \mathrm{ord}_{z^e}(h) \cdot e + \frac{k}{2}(e-1) = \mathrm{ord}_{\pi(z)}(\omega) \cdot e + \frac{k}{2}(e-1)$. For part (2) suppose f has the q-expansion $a_n q^n + \cdots$. Then locally, $\omega = f(\tau)(\mathrm{d}\tau)^{k/2} = a_n q^n \,(2\pi i)^{-k/2} \,\frac{\mathrm{d}q^{k/2}}{q^{k/2}} = a_n q^{n-k/2} \,(2\pi i)^{-k/2} \,(\mathrm{d}q)^{k/2}$. From here we get that $\mathrm{ord}_\tau(f) = \mathrm{ord}_{\pi(\tau)}(\omega) + \frac{k}{2}$. ☐

The elliptic points make the divisor a $\mathbb{Q}$-divisor instead of a $\mathbb{Z}$-divisor, making it so that we cannot apply Riemann-Roch. We fix this by taking $\lfloor \mathrm{div}(f) \rfloor$ instead. Notice that condition $\mathrm{div}(f_0) + \lfloor \mathrm{div}(f) \rfloor \geq 0$ will still work to classify modular forms meaning we still have the isomoprhism $\mathcal{M}_k(\Gamma) \cong L(\lfloor \mathrm{div}(f) \rfloor)$.

Take ω to be the $k/2$-fold differential associated with f. Then $\mathrm{div}(\omega) = \mathrm{div}(f(\tau)(\mathrm{d}\tau)^{k/2}) = \mathrm{div}(f) + (k/2)\,\mathrm{div}(\mathrm{d}\tau)$. If $\{x_{2,i}\}, \{x_{3,i}\},$ and $\{x_{c,i}\}$ are the sets of elliptic points of period 2, elliptic points of period 3, and cusps respectively then by Lemma 3.1 we have

$$\mathrm{div}(\mathrm{d}\tau) = -\sum_i \frac{1}{2} x_{2,i} - \sum_i \frac{2}{3} x_{3,i} - \sum_i x_{c,i}$$

Plugging back in we get

$$\lfloor \mathrm{div}(f) \rfloor = \mathrm{div}(\omega) + \sum_i \left\lfloor \frac{k}{4} \right\rfloor x_{2,i} + \sum_i \left\lfloor \frac{k}{3} \right\rfloor x_{3,i} + \sum_i \frac{k}{2} x_{c,i}.$$

We don't need the floor on the last one because we're taking the case where k is even (even-weight modular forms are more common).

We can check that $\deg(\lfloor \mathrm{div}(f) \rfloor) > 2g-2$. This allows us to apply part (4) from Corollary 3.6.1. Denoting the number of elliptic points of period 2, the number of elliptic points of period 3, and the number of cusps as $\varepsilon_2, \varepsilon_3,$ and ε_∞ repsectively Riemann-Roch finally gives us

$$\dim(\mathcal{M}_k(\Gamma)) = \ell(\lfloor \mathrm{div}(f) \rfloor) = (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \varepsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \varepsilon_3 + \frac{k}{2} \varepsilon_\infty.$$

Similarly the space of cusp forms $\mathcal{S}_k$ can be shown to be isomorphic to $L(\lfloor \mathrm{div}(f) - \sum_i x_{c,i} \rfloor)$. From there applying Riemann-Roch gives

$$\dim(\mathcal{S}_k(\Gamma)) = \ell(\lfloor \mathrm{div}(f) - \sum_i x_{c,i} \rfloor) = \dim(\mathcal{M}_k(\Gamma)) - \varepsilon_\infty.$$

Theorem 3.7 (Dimension Formula)**.** [DS05] *Let Γ be a congruence subgroup, let g denote the genus of $X(\Gamma)$ and let ε_2, ε_3, and ε_∞ denote the number of elliptic points of orders 2 and 3, and the number of cusps, respectively. Then for any even integer $k \geq 2$, the space of modular forms of weight k satisfies*

$$\dim \mathcal{M}_k(\Gamma) = (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \varepsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \varepsilon_3 + \frac{k}{2}\varepsilon_\infty,$$

and the dimension of the space of cusp forms is

$$\dim \mathcal{S}_k(\Gamma) = \dim \mathcal{M}_k(\Gamma) - \varepsilon_\infty.$$

Example 3.8. *The space of cusp forms $\mathcal{S}_{12}(SL_2(\mathbb{Z}))$ has dimension 1, and it is spanned by Δ, the discriminant function.*

4. Hecke Operators

To introduce the idea of a Hecke operator, we must first introduce the *double coset operator*, which takes $\mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2)$. For two congruence subgroups $\Gamma_1, \Gamma_2 \subset SL_2(\mathbb{Z})$ and matrix $\alpha \in GL_2^+(\mathbb{Q})$ the set

$$\Gamma_1 \alpha \Gamma_2 = \{\gamma_1 \alpha \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}$$

is a **double coset**. We take the orbit space $\Gamma_1 / \Gamma_1 \alpha \Gamma_2$, which is the set of $\{\Gamma_1 \beta_j\}$ where the β_j are orbit representatives $\beta_j = \gamma_{j_2} \alpha \gamma_{j_1}$. This means that we can decompose the double coset as the disjoint union of Γ_1 acting on these orbit representatives (cosets of Γ_1): $\Gamma_1 \alpha \Gamma_2 = \bigcup_j \Gamma_1 \beta_j$. One can check that this union is finite.

Before defining the double coset operator we must first extend the weight-k operator to account for matrices in $GL_2^+(\mathbb{Q})$. For some matrix $\beta \in GL_2^+(\mathbb{Q})$, the weight-$k$ operator is defined as

$$f[\beta]_k = (\det \beta)^{k-1}(c\tau + d)^{-k} f(\beta(\tau)).$$

Definition 4.1. *We define the weight-k* ***double coset operator*** *as*

$$f[\Gamma_1 \alpha \Gamma_2]_k = \sum_j f[\beta_j]_k$$

where β_j are orbit representatives.

Proposition 4.2. *The weight-k double coset operator takes $\mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2)$.*

Proof. To show that the double coset operator takes the space of modular forms with respect to Γ_1 to modular forms with respect to Γ_2 we need to show that $f[\Gamma_1 \alpha \Gamma_2]_k$ is weight-k invariant with respect to all $\gamma_2 \in \Gamma$. Since $\Gamma_1 \beta \mapsto \Gamma_1 \beta \gamma_2$ is bijective, $\{\beta_j \gamma_2\}$ is a set of orbit representatives if $\{\beta_j\}$ is. Then

$$(f[\Gamma_1 \alpha \Gamma_2]_k)[\gamma_2]_k = \sum_j f[\beta_j \gamma_2]_k = f[\Gamma_1 \alpha \Gamma_2]_k.$$

□

We now define the Hecke operator, a special case of the double coset operator. There are two types that we want to focus on: $\langle d\rangle$, the *diamond operator* and T_p.

This is the double coset where both subgroups are Γ_1 and $\alpha \in \Gamma_0$. Since $\Gamma_1 < \Gamma_0$, $\alpha^{-1}\Gamma_1(N)\alpha = \Gamma_1(N)$ and there is only one coset, making the only orbit representative α. Furthermore two matrices in Γ_0 with bottom right element $\equiv d \pmod N$ are equivalent up to left muliplication by a matrix in Γ_1. Formally, there is an isomoprhism

$$\Gamma_0(N)/\Gamma_1(N) \cong \mathbb{Z}/N\mathbb{Z}^\times$$

Definition 4.3. *Take $\alpha \in \Gamma_0(N)$ and let $\Gamma_1 = \Gamma_2 = \Gamma_1(N)$. Define the* ***diamond operator*** *$\langle d\rangle$ as the double coset operator $[\Gamma_1\alpha\Gamma_2]_k$. We have have*

$$\langle d\rangle f = f[\alpha]_k$$

where $\alpha \in \Gamma_0(N)$ is any matrix $\begin{pmatrix} a & b \\ c & \delta \end{pmatrix}$ with $\delta \equiv d \pmod N$.

Definition 4.4. *Let $\Gamma_1 = \Gamma_2 = \Gamma_1(N)$ and let $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ where p is prime. Define $T_p = [\Gamma_1\alpha\Gamma_2]_k$ which takes $\mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N))$. After selecting orbit representatives we get the following:*

$$T_p f = \begin{cases} \sum_{j=0}^{p-1} f\left[\left(\begin{smallmatrix} 1 & j \\ 0 & p \end{smallmatrix}\right)\right]_k & \text{if } p \mid N \\ \sum_{j=0}^{p-1} f\left[\left(\begin{smallmatrix} 1 & j \\ 0 & p \end{smallmatrix}\right)\right]_k + f\left[\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix}\right)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\right]_k & \text{if } p \nmid N \text{ and } mp - nN = 1 \end{cases}$$

Something that will become useful in some of the proofs later on will be defining the space of modular forms with a *Dirichlet character* called the *nebentypus.* For a Dirichlet character χ mod N we define the χ-*eigenspace* of $\mathcal{M}_k(\Gamma_1(N))$ as

$$\mathcal{M}_k(N,\chi) = \{f \in \mathcal{M}_k(\Gamma_1(N)) : f[\gamma]_k = \chi(d_\gamma)f,\ \forall\gamma \in \Gamma_0(N)\}$$

where d_γ is the lower right element of γ. The full space of modular forms can be expressed as a direct sum over the eigenspaces or

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_i \mathcal{M}_k(N,\chi_i).$$

This means that if $f = \sum_i f_{\chi_i}$ then $\langle d\rangle f = \sum_i \chi(d) f_{\chi_i}$.

Proposition 4.5. [DS05] *Let $f \in \mathcal{M}_k(\Gamma_1(N))$ and f has Fourier expansion $f(\tau) = \sum_{n=0}^\infty a_n(f)q^n$ then*

$$(T_p f)(\tau) = \sum_{n=0}^\infty (a_{np}(f) + \mathbf{1}_N(p)p^{k-1}a_{n/p}(\langle p\rangle f))q^n$$

and if $f \in \mathcal{M}_k(N,\chi)$ then

$$(T_p f)(\tau) = \sum_{n=0}^\infty (a_{np}(f) + \chi(p)p^{k-1}a_{n/p}(f))q^n$$

where $a_{n/p} = 0$ if $p \nmid n$ and $\mathbf{1}_N(p) = 0$ if $p \mid N$ and 1 otherwise.

Proof. Whether $p \mid N$ or not we have the term $\sum_{j=0}^{p-1} f\left[\left(\begin{smallmatrix} 1 & j \\ 0 & p \end{smallmatrix}\right)\right]_k$. Expanding, we get

$$\begin{aligned} p^{k-1} \cdot p^{-k} \sum_{j=0}^{p-1} f\left(\frac{\tau+j}{p}\right) &= \frac{1}{p} \sum_{j=0}^{p-1} \sum_{n=0}^{\infty} a_n(f) e^{2\pi i n(\tau+j)/p} \\ &= \frac{1}{p} \sum_{j=0}^{p-1} \sum_{n=0}^{\infty} a_n(f) \left(e^{2\pi i \tau/p}\right)^n \left(e^{2\pi i/p}\right)^{nj} \\ &= \frac{1}{p} \sum_{n=0}^{\infty} a_n(f) \left(e^{2\pi i \tau/p}\right)^n \sum_{j=0}^{p-1} \left(e^{2\pi i n/p}\right)^j \end{aligned}$$

The geometric series on the right becomes p if $p \mid n$ and 0 otherwise. Then we will just have

$$\sum_{n \equiv 0 \bmod p} a_n(f) q^{n/p} = \sum_{n=0}^{\infty} a_{np}(f) q^n.$$

If $p \nmid N$ then we also have to deal with the $f\left[\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix}\right)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\right]_k$ term. Notice that $\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix}\right) \in \Gamma_0(N)$. Then this term is equivalent to saying $(\langle p \rangle f)\left[\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\right]_k(\tau)$ which will be

$$p^{k-1}(\langle p \rangle f)(p\tau) = p^{k-1} \sum_{n=0}^{\infty} a_n(\langle p \rangle f) q^{np}.$$

Putting these two together we get the first formula ($\langle p \rangle = 0$ when $p \mid N$). Notice that if $f \in \mathcal{M}_k(N, \chi)$ then $\langle p \rangle f = \chi(p) f$ so the $\chi(p)$ just factors out to give the second one. $\square$

Proposition 4.6. *Let $d, e \in \mathbb{Z}/N\mathbb{Z}$ and let p, q be primes. Then*

(1) $\langle d \rangle T_p = T_p \langle d \rangle$
(2) $\langle d \rangle \langle e \rangle = \langle e \rangle \langle d \rangle = \langle de \rangle$
(3) $T_p T_q = T_q T_p$

These can be easily checked from previous formulas. Now let's look at T_n for n not prime. First let's define T_{p^r}, where p is a prime, recursively as follows:

$$T_{p^r} = T_p T_{p^{r-1}} - p^{k-1} \langle p \rangle T_{p^{r-2}}.$$

From here we can extend the definition to T_n where T_1 is the identity and

$$T_n = \prod T_{{p_i}^{e_i}} \text{ where } p_i \text{ are prime and } n = \prod p_i^{e_i}$$

Notice this means that we have $T_n T_m = T_{nm}$ when $(n, m) = 1$.

Proposition 4.7. [DS05] *Let $f \in \mathcal{M}_k(\Gamma_1(N))$ and f has Fourier expansion $f(\tau) = \sum_{n=0}^{\infty} a_n(f) q^n$ then*

$$(T_n f)(\tau) = \sum_{m=0}^{\infty} a_m(T_n f) q^m$$

where

$$a_m(T_n f) = \sum_{d|(m,n)} d^{k-1} a_{mn/d^2}(\langle d\rangle f)$$

and if $f \in \mathcal{M}_k(N,\chi)$ then

$$a_m(T_n f) = \sum_{d|(m,n)} \chi(d) d^{k-1} a_{mn/d^2}(f)$$

Proof. Since the space decomposes as a direct sum of χ-eigenspaces we only have to consider the second case, as the first case is implied by it. Assume $f \in \mathcal{M}_k(N,\chi)$. We prove this with induction on r with T_{p^r}. Our base case is $r = 1$ because this formula agrees with the formula for T_p. Then since $T_{p^r} = T_p T_{p^{r-1}} - p^{k-1}\langle p\rangle T_{p^{r-2}}$,

$$a_m(T_{p^r}) + \chi(p)p^{k-1} a_m(T_{p^{r-2}}) = a_m(T_p(T_{p^{r-1}} f))$$

Using the Fourier coefficient formula for T_p we get

$$a_m(T_{p^r}) = a_{mp}(T_{p^{r-1}} f) + \chi(p)p^{k-1} a_{m/p}(T_{p^{r-1}} f) - \chi(p)p^{k-1} a_m(T_{p^{r-2}})$$

Assuming the formula holds for $r-1, r-2$ we will have

$$a_m(T_{p^r}) = \sum_{d|(mp,p^{r-1})} \chi(d)d^{k-1} a_{mp^r/d^2}(f) + \chi(p)p^{k-1} \sum_{d|(m/p,p^{r-1})} \chi(d)d^{k-1} a_{mp^{r-2}/d^2}(f)$$
$$-\chi(p)p^{k-1} \sum_{d|(m,p^{r-2})} \chi(d)d^{k-1} a_{mp^{r-2}/d^2}(f)$$

We can rewrite the first term as

$$a_{mp^r}(f) + \sum_{d|(mp,p^{r-1}),\, d>1} \chi(d)d^{k-1} a_{mp^r/d^2}(f).$$

Compare these terms with those in the sum in the subtraction term. Every d' in the first sum will be pd with d in the third sum. If we re-index, this will result in an extra factor $\chi(p)p^{k-1}$ in the first sum, causing the two to cancel. We are left with

$$a_m(T_{p^r}) = a_{mp^r}(f) + \chi(p)p^{k-1} \sum_{d|(m/p,p^{r-1})} \chi(d)d^{k-1} a_{mp^{r-2}/d^2}(f).$$

If we re-index by multiplying every d by p and bring in the first term as the $d = 1$ term then we get our desired form.

Now we set $n = n_1 n_2$ and we have to check $a_m(T_{n_1}(T_{n_2} f))$ where $(n_1, n_2) = 1$. Assuming the formula works for T_{n_1} and T_{n_2} we get

$$a_m(T_{n_1}(T_{n_2} f)) = \sum_{d|(m,n_1)} \chi(d)d^{k-1} a_{mn_1/d^2}(T_{n_2} f)$$
$$= \sum_{d|(m,n_1)} \chi(d)d^{k-1} \sum_{e|(mn_1/d^2,n_2)} \chi(e)e^{k-1} a_{mn_1n_2/d^2e^2}(f)$$

Since $(n_1, n_2) = 1$, we must have that $(d, e) = 1$. Then setting $\ell = de$ we get

$$a_m(T_n f) = \sum_{\ell|(m,n)} \chi(\ell)\ell^{k-1} a_{mn/\ell^2}(f),$$

completing the proof. □

Now we can take a look at an application of Hecke operators. Recall the discriminant function

$$\Delta = g_2^3 - 27g_3^2.$$

This is given the Fourier expansion $\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n$ where τ is the *Ramanujan tau function*. While calculating coefficients of the series Ramanujan noticed a few interesting phenomena. This led him to his conjecture about these values.

Theorem 4.8 (Ramanujan tau conjecture)**.** *Given the modular discriminant function*

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n = q \prod_{n=0}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - \dots$$

(a) $\tau(n)\tau(m) = \tau(mn)$ *if* $(m, n) = 1$
(b) For prime p *and* $j \in \mathbb{N}$ *we have* $\tau(p^{j+1}) = \tau(p)\tau(p^j) - p^{11}\tau(p^{j-1})$
(c) For prime p, $|\tau(p)| \leq 2p^{11/2}$

The third part of the conjecture follows from the Weil conjectures which were proved by Deligne in 1974 [Del74]. This is much more involved and cannot be included in this paper. However, the first two statements were proved by Mordell in 1917 [Mor17], with what are now known as Hecke operators. That is the proof that will be shown.

Proof. Recall that $\Delta(z)$ is a modular form of weight 12 but also notice that the coefficient a_0 in the Fourier series is 0. This means that it is also a cusp form. Recall that the space of weight 12 cusp forms $\mathcal{S}_{12}(SL_2(\mathbb{Z}))$ has dimension 1. We know that the Hecke operator takes $\mathcal{S}_{12}(SL_2(\mathbb{Z})) \to \mathcal{S}_{12}(SL_2(\mathbb{Z}))$. This means that if we apply T_n to this function, we must have $T_n\Delta = \lambda_n\Delta$ because every cusp form in $\mathcal{S}_{12}(SL_2(\mathbb{Z}))$ must be a scalar multiple of Δ. Apply T_n to Δ and see what effect it has on its first coefficient. We get

$$a_1(T_n\Delta) = \sum_{d|(1,n)} d^{k-1} a_{n/d^2}(\langle d \rangle \Delta) = a_n(\Delta).$$

This means that

$$T_n\Delta = \tau(n)q + \cdots .$$

Since this must be a scalar multiple of Δ we must have that $T_n\Delta = \tau(n)\Delta$. Since $T_mT_n = T_{mn}$ if $(m, n) = 1$ and we know that the *eigenvalue* of T_n is $\tau(n)$ we get that $\tau(m)\tau(n)\Delta = T_nT_m\Delta = T_{mn}\Delta = \tau(mn)\Delta$ which implies $\tau(m)\tau(n) = \tau(mn)$ when $(m, n) = 1$.

For the second part let's apply T_p to Δ. If we look at the coefficient of p^r(which is $\tau(p)\tau(p^r)$ because $\tau(p)$ is the eigenvalue) we get

$$\tau(p)\tau(p^r) = a_{p^{r+1}}(\Delta) + p^{11} a_{p^{r-1}}(\langle p \rangle \Delta)$$

Since we're dealing with $SL_2(\mathbb{Z})$, the diamond operator is trivial and is just the identity, meaning that $a_{p^{r-1}}(\langle p\rangle\Delta) = a_{p^{r-1}}(\Delta) = \tau(p^{r-1})$. Rearranging, we get the formula

$$\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1}).$$

□

Since we have these two relations we can now say something about the associated *L-function* with the τ function, which is that its Euler product is given as

$$L(\Delta, s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}.$$

5. Inner Products, Eigenforms, and Newforms

We've already seen that the space of modular forms and the space of cusp forms are both vector spaces. To study the the space of cusp forms further (specifically $\mathcal{S}_k(\Gamma_1(N))$) we want to make it an *inner product space* by defining an inner product operator between two cusp forms.

We first define the hyperbolic measure for $\tau = x + iy \in \mathbb{H}$ as

$$\mathrm{d}\mu(\tau) = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

One can check that acting on τ with a matrix $\gamma \in SL_2(\mathbb{Z})$ doesn't change the differential; in other words $\mathrm{d}\mu(\gamma\tau) = \mathrm{d}\mu(\tau)$. We want to integrate this over the modular curve $X(\Gamma)$ given a congruence subgroup Γ. Let the *volume* of $X(\Gamma)$ be $V_\Gamma = \int_{X(\Gamma)} \mathrm{d}\mu(\tau)$. We define the inner product as the integral $\varphi\,\mathrm{d}\mu(\tau)$ where

$$\varphi(\tau) = f(\tau)\overline{g(\tau)}(\mathrm{Im}(\tau))^k$$

. We're able to integrate this over $X(\Gamma)$ because φ is Γ-invariant.

Definition 5.1. *The **Petersson inner product** $\langle f, g\rangle : \mathcal{S}_k(\Gamma) \times \mathcal{S}_k(\Gamma) \to \mathbb{C}$ is defined by*

$$\langle f, g\rangle_\Gamma = \frac{1}{V_\Gamma}\int_{X_\Gamma} f(\tau)\overline{g(\tau)}(\mathrm{Im}(\tau))^k \mathrm{d}\mu(\tau)$$

Now that we've made the space of cusp forms an inner product space we can define *adjoint operators*. Recall that given a linear operator T and "vectors" u, v the adjoint operator T^* satisfies the condition

$$\langle Tu, v\rangle = \langle u, T^*v\rangle.$$

An operator is considered normal if it commutes with its adjoint ($TT^* = T^*T$). We will now prove some results showing when a Hecke operator is normal and an important consequence of it.

We first need a lemma describing the adjoints of $\langle p\rangle$ and T_p.

Lemma 5.1. [DS05] *For congruence subgroup Γ and matrix $\alpha \in GL_2^+(\mathbb{Q})$ set α' as $\det(\alpha)\alpha^{-1}$. Then*
(a) If $\alpha^{-1}\Gamma\alpha \subset SL_2(\mathbb{Z})$ then for $f \in \mathcal{S}_k(\Gamma)$ and $g \in \mathcal{S}_k(\alpha^{-1}\Gamma\alpha)$,

$$\langle f[\alpha]_k, g\rangle_{\alpha^{-1}\Gamma\alpha} = \langle f, g[\alpha']_k\rangle_\Gamma$$

(b) For $f, g \in \mathcal{S}_k(\Gamma)$,

$$\langle f[\Gamma\alpha\Gamma]_k, g\rangle = \langle f, g[\Gamma\alpha'\Gamma]_k\rangle.$$

Proof. First note that the integral of $\varphi(\alpha(\tau))$ over $\alpha^{-1}\Gamma\alpha\backslash\mathbb{H}$ is the same as the integral of $\varphi(\tau)$ over $X(\Gamma)$ due to a change of variables and the fact that the hyperbolic measure is invariant. We have

$$\int_{\alpha^{-1}\Gamma\alpha\backslash\mathbb{H}} (f[\alpha]_k)(\tau)\overline{(g(\tau))}(\operatorname{Im}\tau)^k \mathrm{d}\mu(\tau) = \int_{\alpha^{-1}\Gamma\alpha\backslash\mathbb{H}} (\det\alpha)^{k-1} f(\alpha(\tau))(j(\alpha,\tau))^{-k}\overline{g(\tau)}(\operatorname{Im}\tau)^k \mathrm{d}\mu(\tau)$$

To switch the integral to be over $X(\Gamma)$ we need to make the transformation $\alpha(\tau) \mapsto \tau$. This is the same thing as $\tau \mapsto \alpha'(\tau)$. We get

$$\int_{X(\Gamma)} (\det\alpha)^{k-1} f(\tau)(j(\alpha,\alpha'(\tau)))^{-k}\overline{g(\alpha'(\tau))}(\operatorname{Im}\alpha'(\tau))^k \mathrm{d}\mu(\tau).$$

Remember that that $\operatorname{Im}(\alpha'(\tau)) = (\det\alpha')\operatorname{Im}(\tau)(j(\alpha',\tau))^{-2}$ and notice that $\det\alpha' = \det\alpha$. We also have that

$$j(\alpha,\alpha'(\tau)) = c(\alpha'(\tau)) + d = \frac{(ca'+dc')\tau + (cb'+dd')}{c'\tau + d'} = \frac{j(\alpha\alpha',\tau)}{j(\alpha',\tau)} = \frac{\det(\alpha)}{j(\alpha',\tau)}.$$

Putting all of these together and simplifying we get

$$\int_{X(\Gamma)} f(\tau)\overline{(g[\alpha']_k)(\tau)}(\operatorname{Im}(\tau))^k \mathrm{d}\mu(\tau).$$

Since the volumes of $\alpha^{-1}\Gamma\alpha\backslash\mathbb{H}$ and $X(\Gamma)$ will be the same, part (a) is proven.

For the second part realize that if β_j are orbit representatives for $\Gamma\alpha\Gamma$ then we can use $\beta_j' = \det(\beta_j)\beta_j^{-1}$ as orbit representatives for $\Gamma\alpha'\Gamma$. Remember that $f[\beta_j]_k$ is invariant on $\beta_j^{-1}\Gamma\beta_j \bigcap \Gamma$ so when we decompose the double coset operator within the inner product we get

$$\langle f[\Gamma\alpha\Gamma]_k, g\rangle_\Gamma = \sum_j \langle f[\beta_j]_k, g\rangle_{\beta_j^{-1}\Gamma\beta_j \bigcap \Gamma}$$

Using part (a) this becomes

$$\sum_j \langle f, g[\beta_j']_k\rangle_{\Gamma \bigcap \beta_j^{\Gamma}\beta_j^{-1}} = \langle f, g[\Gamma\alpha'\Gamma]_k\rangle_\Gamma,$$

which proves part (b). □

Using this we can show the normality of the operators.

Theorem 5.2. [DS05] *For the inner product space $\mathcal{S}_k(\Gamma_1(N))$ and $p \nmid N$ the Hecke operators $\langle p\rangle$ and T_p are normal operators.*

Proof. Recall that $\langle p\rangle$ is $[\alpha]_k$ where $\alpha \in \Gamma_0(N)$ and the bottom right element is $p \pmod N$. Since $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ we know that $\alpha^{-1}\Gamma\alpha = \Gamma$. Then we have $\langle p\rangle^* = ([\alpha]_k)^* = ([\alpha']_k)$. Since $\det(\alpha) = 1$, $\alpha' = \alpha^{-1}$ meaning $([\alpha]_k)^* = [\alpha^{-1}]_k$ so $\langle p\rangle^* = \langle p\rangle^{-1}$ so $\langle p\rangle$ is normal.

For T_p we have

$$T_p^* = [\Gamma_1(N) \left[\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right] \Gamma_1(N)]_k^* = [\Gamma_1(N) \left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right] \Gamma_1(N)]^k.$$

We need to get this as a product of $\left[\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right]$ and other matrices to get it into a form similar to that of T_p. It is true that for $mp - nN = 1$,

$$\left[\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right] = \left[\begin{smallmatrix}1&n\\N&mp\end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right]\left[\begin{smallmatrix}p&n\\N&m\end{smallmatrix}\right]$$

Note that the first matrix in the product is part of $\Gamma_1(N)$ and the third is part of $\Gamma_0(N)$. Substituting and simplifying, we get

$$\begin{aligned}\Gamma_1(N)\left[\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right]\Gamma_1(N) &= \Gamma_1(N)\left[\begin{smallmatrix}1&n\\N&mp\end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right]\left[\begin{smallmatrix}p&n\\N&m\end{smallmatrix}\right]\Gamma_1(N)\\ &= \Gamma_1(N)\left[\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right]\left[\begin{smallmatrix}p&n\\N&m\end{smallmatrix}\right]\left[\begin{smallmatrix}p&n\\N&m\end{smallmatrix}\right]^{-1}\Gamma_1(N)\left[\begin{smallmatrix}p&n\\N&m\end{smallmatrix}\right]\\ &= \Gamma_1(N)\left[\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right]\Gamma_1(N)\left[\begin{smallmatrix}p&n\\N&m\end{smallmatrix}\right].\end{aligned}$$

We can split this into T_p acting on $\left[\begin{smallmatrix}p&n\\N&m\end{smallmatrix}\right]_k$ acting on f. Realize that since $mp - nN = 1$, it must be that $mp \equiv 1 \pmod N$ so $p^{-1} \equiv m \pmod N$. This means that $\left[\begin{smallmatrix}p&n\\N&m\end{smallmatrix}\right]$ is the diamond operator $\langle p\rangle^{-1}$. Thus we have shown that $T_p^* = \langle p\rangle^{-1}T_p$. One can check that this commutes with T_p meaning that T_p is normal. □

Now that we have that the Hecke operators T_p and $\langle p\rangle$ are normal we can invoke the *Spectral Theorem*. Recall the Spectral Theorem from linear algebra states that if there is a family of commuting normal operators on a vector space then there exists an orthonormal basis of simultaneous eigenvectors for all operators in said family. Furthermore, each operator is diagonalizable.

In our case, the commuting normal operators are the Hecke operators and the eigenvectors are called *eigenforms*.

Theorem 5.3. *The space of cusp forms $\mathcal{S}_k(\Gamma_1(N))$ has an orthogonal basis of simultaneous eigenforms for the Hecke operators $\{\langle n\rangle, T_n : (n, N) = 1\}$.*

One thing we can do is look at what happens to forms when we change levels. A simple observation we can make is that if we have $M \mid N$ then $\mathcal{S}_k(\Gamma_1(M)) \supset \mathcal{S}_k(\Gamma_1(N))$. Our goal here is to introduce the idea of *oldforms* and *newforms*. Essentially, oldforms are those forms coming from a lower level and newforms are forms that don't arise from smaller levels.

Definition 5.4. *For $d \mid N$ let φ_d be the map*

$$\varphi_d : \mathcal{S}_k(\Gamma_1(N/d)) \to \mathcal{S}_k(\Gamma_1(N))$$

given by

$$f \mapsto f[\alpha_d]_k$$

where

$$\alpha_d = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$$

.

One can check that for a matrix $\gamma \in \Gamma_1(M)$, $\alpha_d\gamma \in \Gamma_1(N)$. From here we can define the space of oldforms.

Definition 5.5. *For $\mathcal{S}_k(\Gamma_1(N))$, the subspace of oldforms $\mathcal{S}_k(\Gamma_1(N))^{\text{old}}$ is given as*

$$\mathcal{S}_k(\Gamma_1(N))^{\text{old}} = \{\varphi_d(f) \text{ for } d \mid N, d \neq N, f \in \mathcal{S}_k(\Gamma_1(N/d))\}$$

The new space $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ is the orthogonal complement of the oldforms. Formally,

$$\mathcal{S}_k(\Gamma_1(N))^{\text{new}} = \{f \in \mathcal{S}_k(\Gamma_1(N)), \langle f, g\rangle = 0 \text{ for all } g \in \mathcal{S}_k(\Gamma_1(N))^{\text{old}}\}$$

We can now make the definition for an eigenform.

Definition 5.6. *A modular form $\mathcal{M}_k(\Gamma_1(N))$ that is an eigenform for the operators T_n and $\langle n\rangle$ is a* ***Hecke eigenform****. When $a_1(f) = 1$, it is called a* ***normalized eigenform****. A* ***newform*** *is a normalized eigenform in $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$.*

Furthermore, because of these eigenvalues we get a general statement about L-functions of these normalized newforms.

Theorem 5.7. *Given normalized newform $f \in \mathcal{S}_k(N, \chi)$, where $f = \sum_{n=0}^{\infty} a_n q^n$, the associated L-function $L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s}$ has Euler product expansion*

$$L(f, s) = \prod_p (1 - a_p(f)p^{-s} + \chi(p)p^{k-1-2s})^{-1}$$

Proof. Since $f \in \mathcal{S}_k(N, \chi)$ is a normalized newform, it is a Hecke eigenform with multiplicative Fourier coefficients:

$$a_{mn} = a_m a_n \quad \text{if } (m, n) = 1, \quad \text{and} \quad a_{p^r} = a_p a_{p^{r-1}} - \chi(p)p^{k-1}a_{p^{r-2}}.$$

This recurrence matches the expansion of

$$\left(1 - a_p p^{-s} + \chi(p)p^{k-1-2s}\right)^{-1} = \sum_{r=0}^{\infty} a_{p^r} p^{-rs}.$$

By multiplicativity, we obtain the Euler product:

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left(1 - a_p p^{-s} + \chi(p)p^{k-1-2s}\right)^{-1}. \qquad \square$$

6. Correspondences and the Eichler-Shimura Relation

Now let's focus on Hecke operators' geometric aspect through *correspondences.* [RS17]

Definition 6.1. *Given curves X and X' a* ***correspondence*** *$T : X \rightsquigarrow X'$ is a pair of surjective morphisms α and β such that*

$$X \xleftarrow{\alpha} Y \xrightarrow{\beta} X'$$

We will think about Hecke operators as Hecke correspondences $T_p : X_0(N) \rightsquigarrow X_0(N)$ where $p \nmid N$. The diagram for this correspondence will be

$$X_0(N) \xleftarrow{\alpha} X_0(pN) \xrightarrow{\beta} X_0(N)$$

The curve $X_0(N)$ can be viewed as a set of isomorphism classes for elliptic curves (called a *moduli space*), specifically pairs (E, C) where E is an elliptic curve and $C \subset E$ is a cyclic subgroup of order N. Likewise,

$X_0(pN)$ classifies isomorphism classes of (E, G) where $G = C \oplus D$ and $D \subset E$ is a cyclic subgroup of order p. Since $(p, N) = 1$ we know that G is a cyclic subgroup of order pN. We can define the maps α and β as

$$\alpha : (E, G) \to (E, C)$$

$$\beta : (E, G) \to (E, (C + D)/D)$$

The map α is given by $\Gamma_0(pN)\backslash\mathbb{H} \to \Gamma_0(N)\backslash\mathbb{H}$ (note that it is a subset so it's the identity map). Likewise the map β is given by the isomorphism

$$\Gamma_0(pN)\backslash\mathbb{H} \cong \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(N) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \backslash\mathbb{H}$$

which sends $z \mapsto \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} z = pz$ due to conjugation. Both α and β induce *pullback maps* on 1-forms. These are just taking a function from one space and putting onto another via composition. In this case it is $X_0(N)$ and $X_0(pN)$. α gives us the pullback maps

$$\alpha^* : H^0(X_0(N), \Omega^1) \to H^0(X_0(pN), \Omega^1).$$

where $H^0(X, \Omega^1)$ is just the space of 1-forms on X (called the sheaf cohomology group). Recall that weight k modular forms correspond to $k/2$-fold differential forms, meaning we can identify $\mathcal{S}_2(\Gamma)$ with $H^0(X(\Gamma), \Omega^1)$. This means that α^* is taking $\mathcal{S}_2(X_0(N)) \to \mathcal{S}_2(X_0(pN))$. Then we define T_p as the map that first pulls back with α^* and then pushes forward with β_* giving us $T_p = \beta_* \circ \alpha^*$. We define the pushforward

$$\beta_* : H^0(X_0(pN), \Omega^1) \to H^0(X_0(N), \Omega^1)$$

to be the unique differential $\beta_*(\omega) \in H^0(X_0(N), \Omega^1)$ given $\omega \in H^0(X_0(pN), \Omega^1)$ such that

$$\int_\gamma \omega = \int_{\beta^{-1}(\gamma)} \beta_*(\omega)$$

for all paths γ.

Recall the idea of a *divisor* from before. It's true that the operator T_p also gives us a map between divisors. Say that the *degree* of a map α is the number of points in the preimage for a generic point in the image (for example the map $x \mapsto x^2$ has degree 2). Say that α has degree d. The pullback map α^* on a divisor $D = \sum_i n_i \cdot x_i$ will give $\alpha^*(D) = \sum_i n_i \left(\sum_j e_j \cdot z_j \right)$ where z_j is in the preimage of α and e_j is the multiplicity of that point. Then we see that this takes degree n divisors on X to degree dn divisors on Y. If we apply β_* to this we get a divisor on X'. If we set $n = 0$, only focusing on degree 0 divisors on X we get a map to degree 0 divisors on X'. Going back to T_p we see that

$$T_p : \mathrm{Div}^0(X_0(N)) \to \mathrm{Div}^0(X_0(N)).$$

Furthermore this induces a map on Jacobians, which are *abelian varieties*. The Jacobian of a curve is the quotient

$$J(X_0(N)) = J_0(N) = \frac{\mathrm{Div}^0(X_0(N))}{\text{principal divisors}},$$

where principal divisors are divisors of the form $\mathrm{div}(f)$ for some function f. We know from Proposition 3.2 that for compact Riemann surfaces these always have degree 0. We get

$$T_p : J_0(N) \to J_0(N).$$

This is an *endomorphism* (map to itself). To understand the *Eichler–Shimura congruence relation*, we must study how this map behaves when we reduce modulo a prime $p \nmid N$. It is true $X_0(N)$ has *good reduction at* p when $p \nmid N$. It turns out that you can consider $J_0(N)$ as an abelian variety over the finite field $\mathbb{F}_p$. Note that this has *characteristic* p (we have to add the multiplicative identity p times to get the additive identity). To state the relation we need to introduce the *Frobenius endomorphism* denoted as "Frob" and its transpose *Verschiebung* denoted as "Ver".

Definition 6.2. *For an abelian variety of characteristic* p *the operator* Frob_p *is induced by the map* $x \mapsto x^p$*.*

In our case the variety in question $J_0(N)_{\mathbb{F}_p}$. Now we can state the Eichler-Shimura Congruence Relation.

Theorem 6.3 (Eichler-Shimura Congruence Relation)**.** *Let* N *be a positive integer let* $p \nmid N$ *be a prime. Then*

$$T_p = \mathrm{Frob}_p + \mathrm{Ver}_p \in \mathrm{End}(J_0(N)_{\mathbb{F}_p})$$

Remark. The Eichler–Shimura relation implies that, for a newform $f \in S_2(\Gamma_0(N))$, the eigenvalue of the Hecke operator T_p equals the trace of Frobenius acting on a certain 2-dimensional ℓ-adic Galois representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$

attached to f [Del71]. That is,

$$\mathrm{tr}(\rho_f(\mathrm{Frob}_p)) = a_p(f) \text{ and } \det(\rho_f(\mathrm{Frob}_p)) = p$$

where $a_p(f)$ is the p-th Fourier coefficient of f. In this way, the arithmetic of modular forms becomes encoded in the Galois action on torsion points of Jacobians of modular curves.

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