Fair Division

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Fair Division

This is the distribution of goods or burdens in a way that seems fair to everyone.

Hugo Steinhaus is credited with the introduction of the fair division problem in a modern mathematical way.

Valuation

People do not value things the same way. If one person splits something in half that they perceive as half, the other person may see this split 25% and 75%.
We can say that for each person i, they have a function $f_i(x)$ that gives the worth of object x.

Criteria

To define fair mathematically, we can use some criteria the division has to follow.

1. Proportional
    1. For n people, each person gets $1/n$ of the total value
    2. $f_i(x) \geq 1/n$
2. Envy-Free
    1. Each person wants other people's pieces less than or equal to their own piece.
    2. If x is someone's piece, and y is someone else's, $f_i(x) \geq f_i(y)$

Vocab

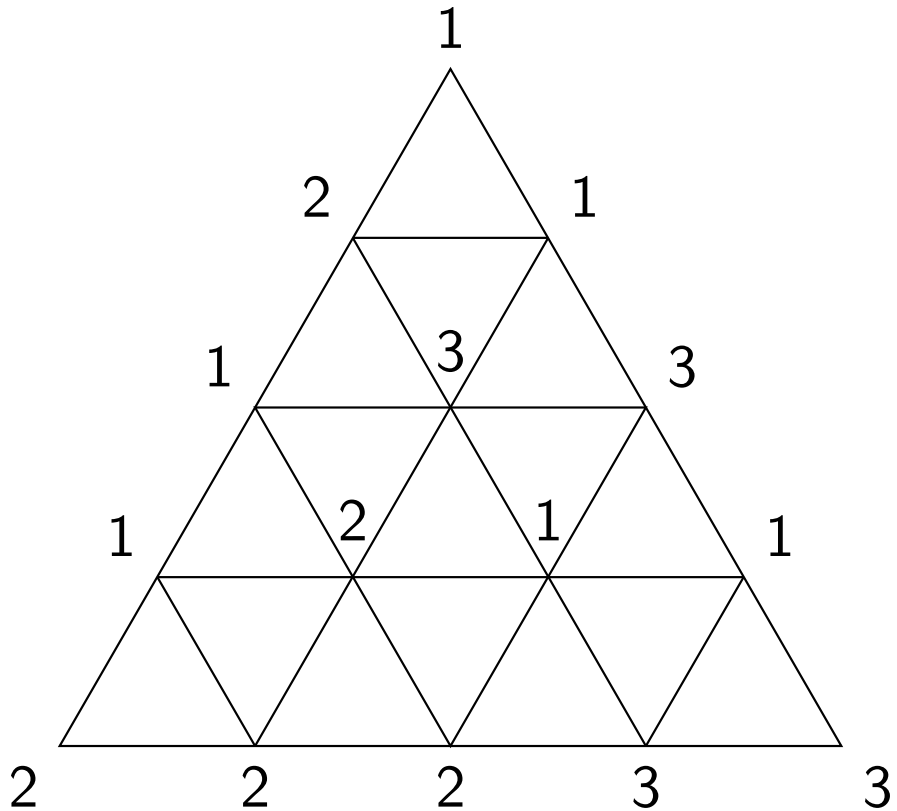
Definition

A *Simplex* is a generalization of a triangle or tetrahedron to a certain number of dimensions.

Definition

A *Triangulation* is the division of a simplex into smaller simplices that intersect shared faces, and the union of all simplices covers the entirety of the original simplex.

Sperner's Labeling

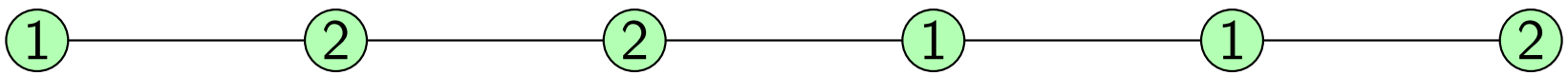


Sperner's Lemma

Lemma

Sperner's Lemma: For any triangulated simplex with boundary-respecting vertex labeling, there exists at least one fully labeled subsimplex (i.e. a simplex where each vertex has a distinct label from the original simplex's set).

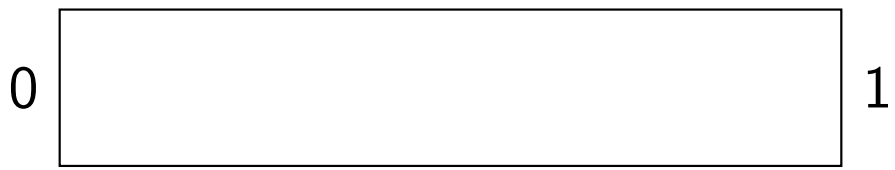
Sperner's Lemma for 1-D



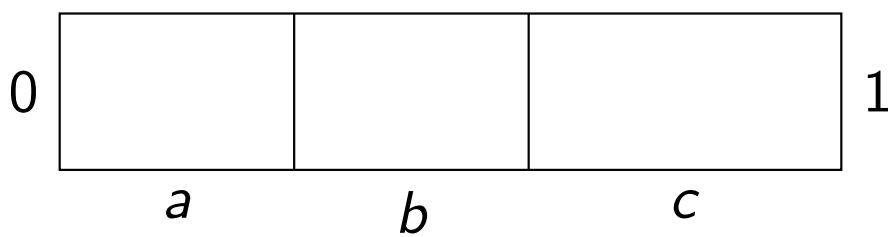
The number of times that the number at a vertex switches will be odd, because the label eventually changes from 1 to 2. The lowest whole odd number is 1, so the minimum times that the labeling switches is 1. Since the labeling is guaranteed to switch, there will be a fully labeled subsimplex.

Cake-cutting Problem

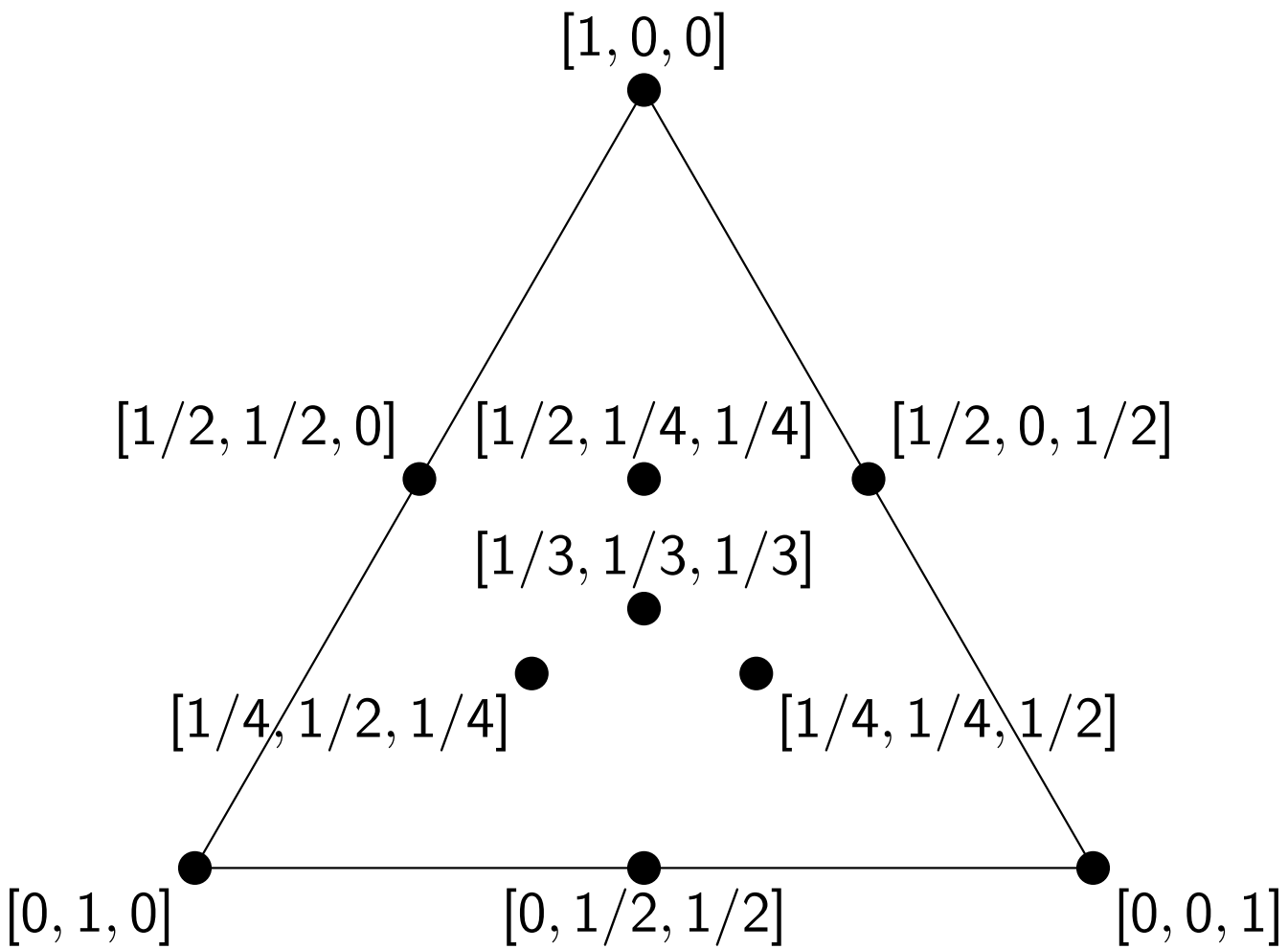
We can represent the cake we need to split for 3 people as the interval $(0, 1)$



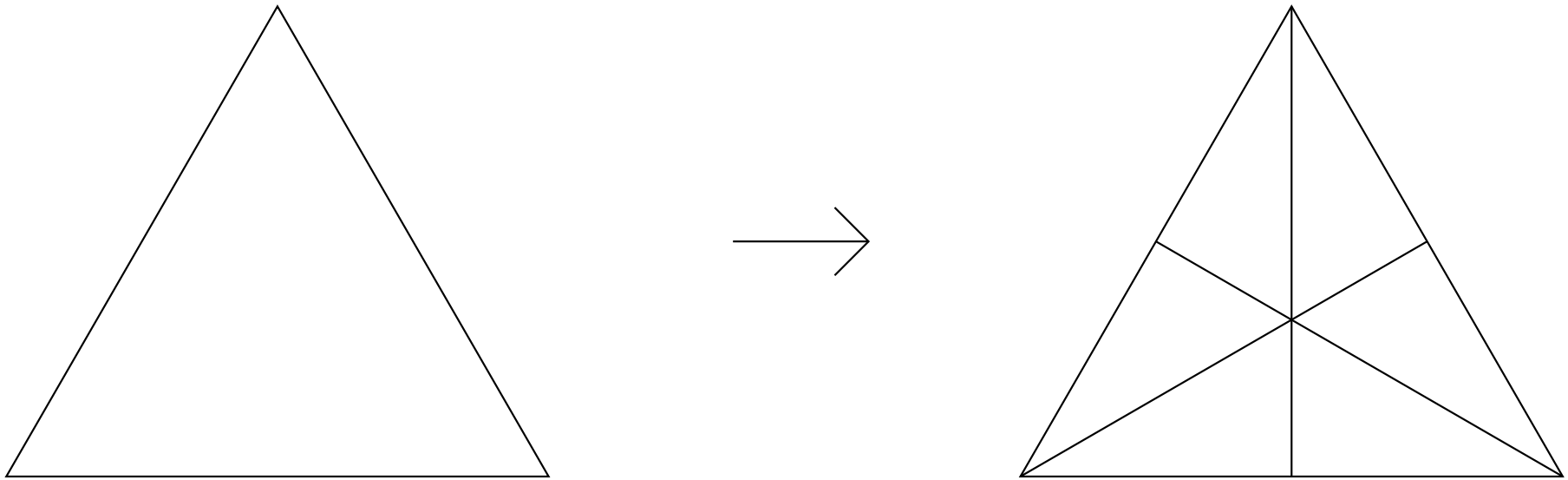
and we can represent a division of the cake as (a, b, c), where the cake is split up into the intervals $(0, a)$, $(a, a + b)$, and $(a + b, 1)$, as a point inside of a triangle.



Barycentric Coordinates



Barycentric Subdivision



Labeling the Triangulation

Since every point inside the triangle represents a division, we can go to each point and ask each person what they value the piece that they receive, or use their function $f_i(x)$. Then we label the point with the name or number that corresponds the the person who values the piece the most.

Connection to Sperner's Lemma

Before we can use Sperner's Lemma, we have to make sure the labeled triangulation is a Sperner's Labeling.
At the main vertices, one person gets the whole cake, so they get labeled there. Alongside the edges, two people get some of the cake, and the last person does not. The labeling will be one of the two people who get the cake.

Using Sperner's Lemma

Now we can use Sperner's Lemma. This means there is a fully labeled triangle somewhere in the simplex. Since the simplex was subdivided into many small subsimplices, the variation between the coordinates of the vertices are very small. This results in a fair division.