FAIR DIVISION PROBLEM

SOMIL SARODE

Contents

1. Abstract

The fair division problem is about the equal distribution of goods or burdens among people based on criteria for fairness such as proportionality and envy-freeness. These concepts are fundamental to solving real-world issues like inheritance disputes, rent division, and chore assignments.

This paper explores five aspects of fair division, starting with a run-through of the proof of Sperner's Lemma. We will then demonstrate its application in the classic cake-cutting problem, establishing the existence of envy-free divisions of divisible goods. Building on this, we examine various methods and complications involved in dividing general divisible goods, especially as the number of participants increases. Next, we address the complexities of

Date: July 14, 2025.

dividing undesirables, such as chores, where fairness requires reversing traditional preference structures and adapting existing protocols. Finally, we explore the division of indivisible goods, which calls for discrete algorithms and approximations, often lacking the guarantees of a fair division provided by tools such as Sperner's Lemma.

2. Introduction

Since the dawn of civilization, humans have always had the problem of dividing things such as responsibilities and resources in a way that is perceived as fair to everyone. In the current world, this problem takes shape in smaller cases such as dividing inheritence and chores among roommates, but it can also appear in an international setting where there is a land conflict between two countries.

One of the first forms of fair division of goods between two people comes in the form of the *divide and choose* method. This method has a person who divides the goods or chores. This division is proposed to the second person, who gets to choose which half of the goods or chores they want.

Before we talk about why this works, it is important to understand that people value different things differently. This means that something that makes up less of the total material could be valued a lot by a person if they really like it. An example of this is a diamond in a lot of dirt. Although the diamond may make up less than 1 percent of the mass, it is easily worth more than the rest of the dirt.

So, since we now know that the cost of something is subjective, let us talk about the *divide and choose* method. This method would usually lead to the first person splitting the goods or chores in such a way that each half is equivalent. "Why", you may ask. This is because if the first person does not do that, the second person will take the better half, ultimately leaving the first person with the worse half. In the first person's eyes, they would make it so that each half would be worth 50 percent of the total, but for the second person, since they get to choose which half they want, they can always get the half that they believe is worth 50 percent or more of the total (remember, people value different things differently). This leads to fairness, as it fills the two fair division criteria which are:

(1) Proportionality
(2) Envy-Freeness

So what do proportionality and envy-freeness mean?

Definition 2.1. Proportionality means that each person feels like they have received a fair piece, which is defined in this case as receiving what they perceive as $1/n$ or more of the total share, with n being the number of pieces made.

Definition 2.2. Envy-Freeness means that each person wants their own piece more than or equal to a piece that another person received.

We can check that the *divide and choose* method satisfies these two criteria of fairness from both points of view.

For the first person, regardless of the case, they receive 50 percent, also known as $1/2$, fulfilling the proportionality criteria. They also like each piece equally, so the envy-freeness criteria is also fulfilled.

For the second person, they have to choose which half they take, so they will obviously take the half that they value more. Since the goods or chores are divided into two, the halves will be equal, or they will be unequal, leading one to be more than 50 percent, and the other will be less. The second person can always get 1/2 or more of the cost of the goods that satisfy the proportionality criteria. Since the second person can also choose which half they want, they will always like their piece more than or equal to the other piece, fulfilling the envy-freeness criteria.

The actual study of the fair division problem was started in the early 1940s by the Polish mathematician Hugo Steinhaus. He worked on this problem alongside Stefan Banach and Bronislaw Knaster to develop procedures such as *Knaster Inheritance Procedure*, which uses sealed bids and money to help fairly split items that are not indivisible throughout.

Later, during the 1960s and 1970s, topological theorems such as *Sperner's Lemma*, *Brouwer's Fixed Point Theorem*, and the *Borsuk-Ulam Theorem* were used in the proofs to guarantee the existence of a fair division solution under certain conditions. Although the existence of a solution was proven under certain conditions, sometimes there was no method to compute these solutions.

In the 1990s, Steven Brams and Alan Taylor introduced *The Adjusted Winner Procedure*, which uses point-based bidding to divide things fairly between two people, and they also worked on generalizing the envy-free cake cutting problem to cases where there are more than two people.

3. Proving *Sperner's Lemma*

3.1. **Key Terms and Concepts.** What does *Sperner's Lemma* even say in the first place? But first, there is some vocabulary that we should learn beforehand.

Definition 3.1. A *Simplex* is a generalization of a triangle or tetrahedron to a certain number of dimensions. For example, a $0-simplex$ is a point, a $1-simplex$ is a line segment, a $2-simplex$ is a triangle and a $3-simplex$ is a tetrahedron.

Definition 3.2. A *Triangulation* is the division of a simplex into smaller simplices (plural of simplex) that intersect shared faces, and the union of all simplices covers the entirety of the original simplex.

Definition 3.3. A $n-simplex$ is considered *Sperner's Labeled* if each vertex is labeled by one of the $n+1$ distinct labels. Every vertex on a face of the original simplex is only labeled with labels from the vertices of that face.

For example, as you can see with the $2-simplex$ in Figure 1, the vertices are labeled with labels $1,2$, and 3. The edges with vertices labeled 1 and 2 can only be labeled using the numbers 1 and 2. The same is true for the edges with the labels 2 and 3, and with the labels of 1 and 3. Vertices on the inside of the simplex can be labeled with any label that corresponds to the labels on the main vetrices of the simplex.

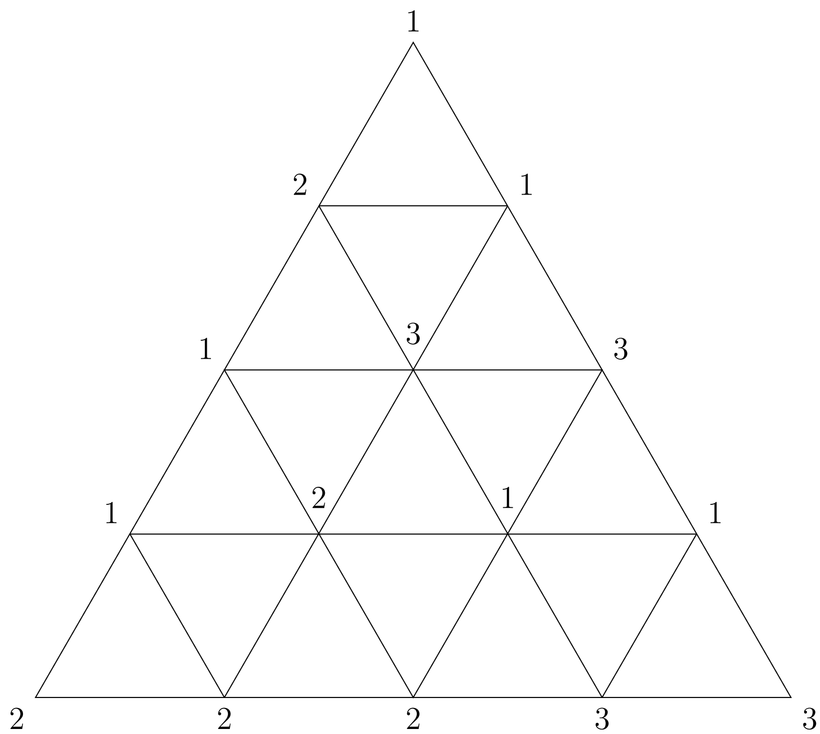

Figure 1

Lemma 3.4 (Sperner's Lemma)**.** *For any triangulated simplex with boundary-respecting vertex labeling, there exists at least one fully labeled subsimplex (i.e. a simplex where each vertex has a distinct label from the original simplex's set).*

3.2. ***Sperner's Lemma* for a** $1-simplex$**.** Let us prove this inductively, using $1-simplex$ at first and transitioning to $2-simplex$. In the $1-simplex$ below, the endpoints are labeled 1 and 2. This means that any vertices in the simplex must be labeled either 1 or 2, as seen below in Figure 2.

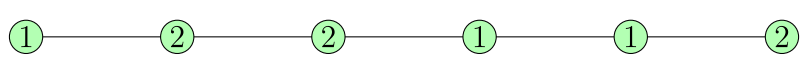

Figure 2

At some point, there will be a fully labeled subsimplex because the parity is odd. When going from left to right along the simplex, the label will change an odd number of times because it starts with the label 1, and has to change at some point to get to the label 2. Once one of the vertices is labeled 2, there will be an even amount of changes, since it has to change to 1, and then back to 2. Since an even number plus one is always odd, this means that the number of swaps is odd, otherwise meaning that the parity is odd. This guarantees that there is a fully labeled $1 - simplex$, as the smallest odd positive whole number is 1.

3.3. ***Sperner's Lemma* for a** $2 - simplex$**.**

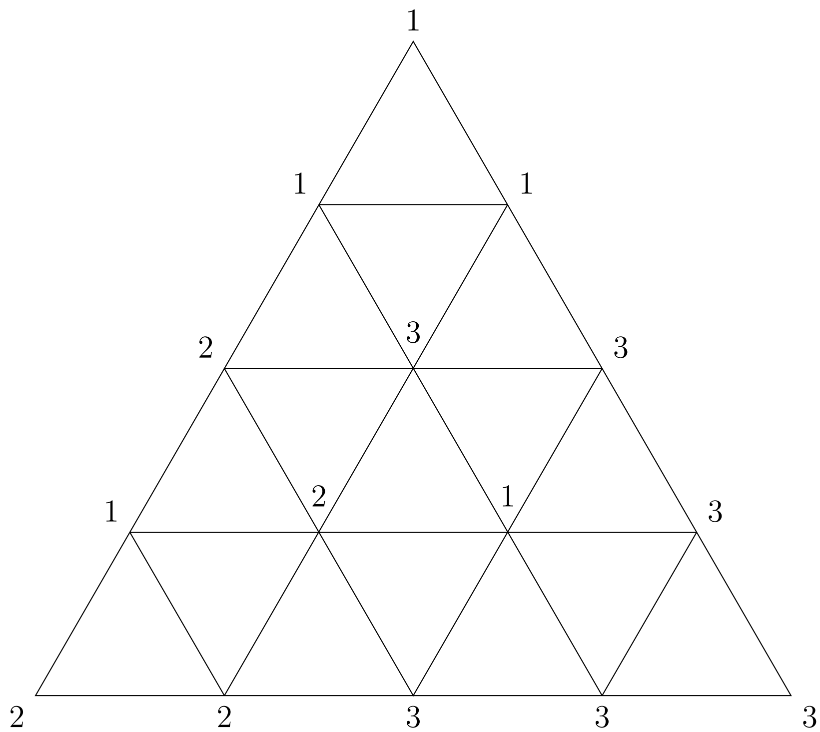

Figure 3

Now, looking at the 2-simplex version, we will initially label it so that it is *Sperner's Labeled*. This is shown in Figure 3. We are trying to show that there will be at least one subsimplex with its vertices labeled 1, 2, and 3. An important thing to remember here is that if a subsimplex has an edge labeled with 1 and 2, and if it also has a vertex labeled with 3, that means that it is a subsimplex that is fully labeled, which leads to the conditions for *Sperner's Lemma* to be met.

Now in Figure 4, all edges labeled with 1 and 2 are in bold and should be easy to see. The graph has also been converted to a dual graph, which means that each subsimplex also contains a node. Everything outside of $2 - simplex$ is also considered as one node. We also draw a line connecting two nodes if they share an edge labeled 1 and 2. If a subsimplex only has 1 edge labeled with a 1 and a 2, this will mean it is fully labeled as the last vertex has to be labeled with the number 3. This means the node corresponding to a fully labeled triangle has an odd degree.

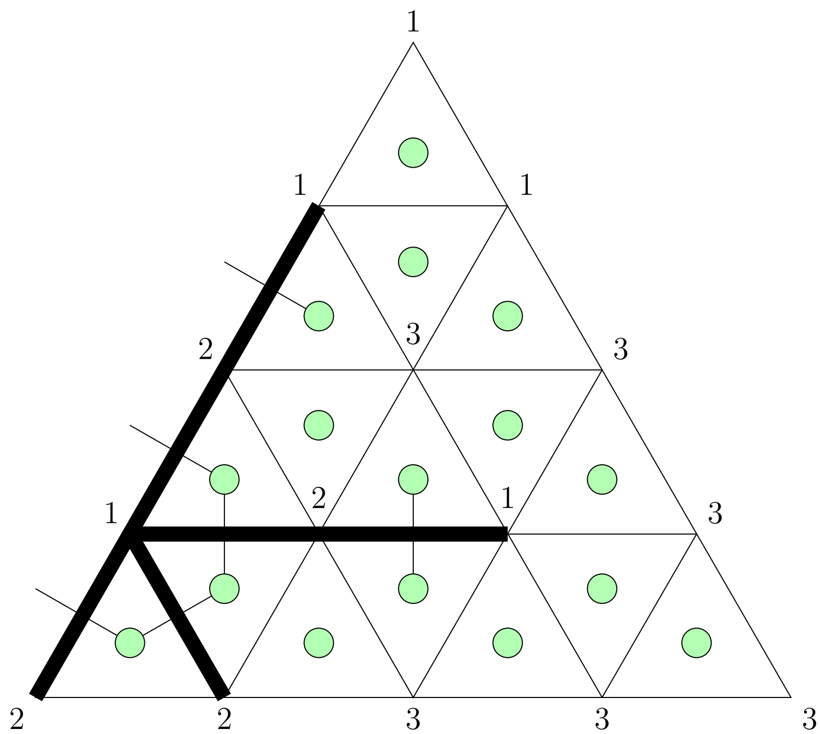

Figure 4

Definition 3.5. The *degree* of a node is defined as the number of edges connected to the node. For example, if a node has 3 edges connected to it, it has a degree of 3.

From the proof of *Sperner's Lemma* for a $1-simplex$, we know that there will be an odd number of edges labeled with 1 and 2 along the main $2-simplex$. In the case shown in Figure 4, there are 3 edges that go to the area outside of the simplex. This means the node representing the area outside of the simplex has an odd degree, as the number of edges connecting it is odd. So now what do we do with this information?

Corollary 3.6. *A corollary to the* Handshaking Lemma *states that in any finite undirected graph, the number of vertices with an odd degree is always even.*

Since we already have one node that has an odd degree, there must be an odd number of other nodes that also have an odd degree. Since the smallest whole odd number is 1, there must be at least one other node with an odd degree. Since the nodes are representing the subsimplexes, and the degree represents the number of edges labeled with a 1 and a 2, there must be a subsimplex that has an odd number of edges labeled with a 1 and a 2. The only amount of odd edges there can be is one, so this shows that there is a fully labeled subsimplex, thus proving *Sperner's Lemma*.

4. Proving the Cake-cutting Problem Using *Sperner's Lemma*

Now we move on to one of the more well known sides of the fair division problem which takes shape in the cake cutting problem. This problem makes the cake heterogeneous, which means it has different flavors throughout the cake. The cake has different flavors, so that means that different people will value different parts of the cake more than other parts due to the fact that they like a certain flavor more than others. Since the thing being split is a cake, that means you can also split it anywhere you want, making it a divisible good.

We can show that a division always exists using *Sperner's Lemma*, which we discussed in the last section. This method shows the existence of a solution to the problem, but it is not a method that shows how to get to the solution to the problem.

First, the cake will be represented by the interval $[0, 1]$. Different intervals will represent different parts of the cake. For example, the interval $[0.5, 1]$ will represent the second half of the cake. We will prove the existence of a solution for a case with 3 people. To prove for more and more people, it is necessary to use higher order simplexes.

To use *Sperner's Lemma*, we first need to find a way to turn the interval into a triangulation. One way we can do this is by turning our divisions into points inside of the simplex. For each point, we can use (a, b, c), where $a + b + c = 1$. One person will receive the part of the cake represented by $[0, a]$, the next will receive the part represented by $[a, a + b]$, and the last person will receive the part represented by $[a + b, 1]$, which can also be written as $[1 - c, 1]$.

A question you may have is how can we represent a two-dimensional point in a triangle with three coordinates. This uses a system called *Barycentric Coordinates*. So, how does this system work? Each number represents a mass placed at the vertex of the triangle. The point is found from these numbers by finding the center of mass. Some examples of points are $[0, 0, 1]$, $[0, 1, 0]$, $[1, 0, 0]$, $[1/3, 1/3, 1/3]$, where the first three points represent the vertices of the triangle, and the fourth point is the centroid of the triangle. More examples of barycentric points can be found in Figure 5.

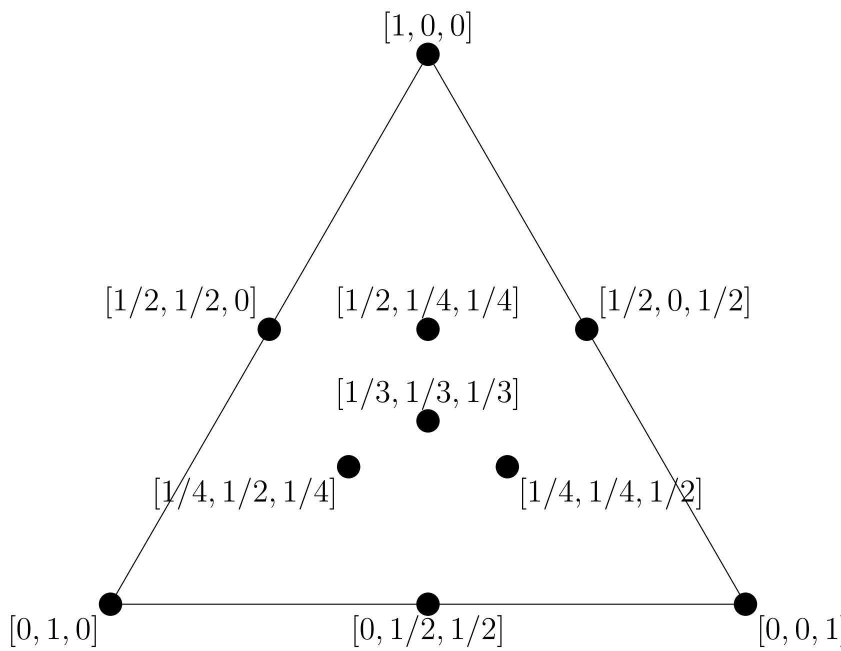

Figure 5

Now that we have a way to represent the divisions of the cake as points in the triangle, we can subdivide the triangle into smaller triangles. To do this method, you have to subdivide the triangle many times. We will subdivide the triangle using *Barycentric Subdivision*.

With a triangulation now in place, we can go to each vertex and propose a division to each person. We then ask each person who prefers their piece the most, or in other words, who would pay the most for their piece. The vertex is then labeled with the person's name or something that marks the vertex that can be connected to them.

How does this guarantee the specific conditions for a *Sperner's Labeling*, you might ask.

We can then use *Sperner's Lemma* since the triangle meets the restrictions. This guarantees that there is a triangle that is fully labeled, which means that if you shift the division cuts a little bit, a different person will prefer their piece the most. This means that each person values their piece of the cake roughly equally, fulfilling the proportionality criteria of fair division. Since each person also prefers a different piece, this means that the division is also envy-free, as nobody wants another person's piece.

This method works to prove the existence of a solution as it goes over every possible division. This is also the reason why it is not feasible to use this method to find an exact solution, as the amount of time you need to check each solution and ask each person for their preference is astronomical.

For a generalization for n people, a $n-1-simplex$ needs to be used. Although complexity skyrockets with each dimension, *Sperner's Lemma* guarantees the existence of a fair division by guaranteeing a fully labeled subsimplex.

5. Division of Divisible Goods

I know that I talked a little bit about divisible goods earlier, but I did not give a concrete definition.

Definition 5.1. A *divisible good* is something that can be split into smaller parts without losing value or functionality.

Some examples of a divisible good include the cake example from earlier, or other things such as land, water, and resources such as gasoline and wood.

How can we model this? Let C represent a cake or divisible good and is represented by the interval $[0, 1]$, and let there be n people, who each have their own function f_i, where $f_i([a, b])$ quantifies how much they value the cake or divisible good they receive, and how much they want piece $[a, b]$ for person i. Some basic restrictions and properties with this function is that f_i is that it is:

- **Non-negative:** $f_i([a, b]) \geq 0$,
- **Additive:** $f_i([a, c]) = f_i([a, b]) + f_i([b, c])$ for $a < b < c$,
- **Normalized:** $f_i([0, 1]) = 1$.

Do you remember the the the two criteria for a fair division? The two criteria were to be proportional in terms of value and make it so that no person envies the other. These criteria reflect different notions of fairness. Depending on the context—legal, economic, or computational—a different criteria may be weighted more.

5.1. Theoretical and Algorithmic Methods.

Divide and Choose: As stated in the introduction, this was one of the first ways to divide something fairly.

***Sperner's Lemma*-Based Existence Proof:** One fundamental way to prove that there is a fair division is with *Sperner's Lemma*.

Moving-Knife Procedure: Moving-knife procedures are continuous protocols that were developed to provide fair divisions with real-time interaction. One of the earliest and most well-known is the Dubins–Spanier method, which values proportionality. In this method, a knife moves slowly across the cake from left to right. Each person (n people in total) monitors the value of the left segment of the cake. When the first person believes the knife has reached a point where the left portion is worth exactly $1/n$ of the cake (according to their valuation), they tell whoever is moving the knife to stop. The person then gets to receive that portion. The process continues recursively on the remaining cake for the other people. This method does not always ensure that the division will be envy free, as the last person does not get a say in the piece they get, and they might want someone else's piece of cake. It also relies on people being trustworthy and being truthful to their evalulations.

There are other more complex procedures have been developed to guarantee envy-free outcomes for multiple people. These methods often involve multiple knives, trimming, and intricate conditions to preserve fairness no matter want another person does. One of these procedures is called Austin's Moving Knife Procedure.

Austin's Moving Knife Procedure guarantees envy-freeness, but it only works for three people. For this method, the first person divides the cake into three pieces. They divide it into such a way that they value each piece equally. After this, the other two people assess the divisions, and if the two other people want the same piece, the first person keeps on moving the cuts, but in such a way that each piece is still valued the same to them.This goes on until the two other people want different pieces, and then they take the piece they want, and the first person takes the leftover piece. One of the requirements for the procedure to work is the assumption that all three people have different preferences.

Though other moving knife procedures theoretically work for any amount of people, they are difficult to extend to larger numbers of people in a way that preserves envy-freeness or proportionality. Additionally, they require continuous monitoring, which sometimes makes it impractical if you can not communicate in real time.

Discrete Approximation using Triangulation: To move toward algorithmic implementations, continuous models can be discretized. When a model gets discretized, it means that the item is divided into a large number of small pieces. The next step is to let the people that the item is being shared between to then assign a value or rank each of these small pieces.

The goal of this process is to group these small pieces in such a way that each person's bundle satisfies a chosen fairness criterion. The triangulation approach from Sperner's Lemma can also be applied in a discrete fashion, enabling algorithms to approximate envy-free or proportional solutions that are close enough to fair for many people.

Discrete methods are especially important for computational fair division because they allow for finite calculations, and they usually use things like a table of preferences to help figure out how the item should be divided.

Algorithmic Cake-Cutting Protocols: There have been many protocols that have been developed to have fair divisions through deterministic algorithms. For instance:

- The **Even–Paz Protocol** provides a proportional division for any number of people in $O(n \log n)$ time. It works by recursively dividing the cake and asking people to determine their worth. It is very similar to the discrete approximation using trigulation, as it splits up the cake. The difference is that the Even-Paz Protocol sacrifices some aspects of fairness to make the protocol applicable to real life.

- The **Selfridge–Conway Procedure** achieves envy-freeness for three people by shaving small pieces off big pieces and redistributing them. It is complex, but it remains one of the most widely discussed envy-free algorithms when n is fairly small.
- **Round-Robin with Trimming**: people take turns selecting their most preferred piece. When someone starts to feel envious of a piece that another person got, adjustments to fix this such as trimming or trading pieces are used to correct imbalances.

These algorithmic approaches are practical and well-suited to software implementations, though they may sacrifice stronger fairness criteria when generalized to larger numbers of people.

5.2. **Choosing a Method.** Choosing the appropriate method depends on the application context, number of people, and the desired fairness criteria. A brief guide:

Context	**Recommended Method**
Existence proof (theoretical)	Sperner's Lemma
Two or three people	Moving-knife or Selfridge–Conway
Many people (large n)	Even–Paz or discrete triangulation
Software or simulations	Discrete approximation or round-robin algorithms

5.3. **Real-World Applications.** The division of divisible goods appears in many settings in real life:

- **Inheritance Division:** Splitting inherited land or money among family members.
- **Time-Sharing:** Assigning lab time, telescope access, or shared computing resources.
- **Media Advertising:** Dividing airtime slots on radio or TV across competing clients.
- **Academic Scheduling:** Allocating classroom hours or equipment fairly among instructors.

6. Division of Undesirable Goods

While a lot of the fair division papers and sections in books focus on dividing desirable and divisible goods, a significant and often more complex challenge occurs when the items to be divided are undesirable. This includes chores, debts, costs, burdens, or any other negative thing that people want to avoid. In these cases, the goal is to allocate the least unfair share to each person. This is basically the reverse of the cake-cutting problem.

6.1. **An Undesirable Problem.** Let C denote a chore or the undesirable good being divied up, represented by the interval $[0, 1]$, and let there be n people, who each have their own function d_i, where $d_i([a, b])$ quantifies how annoying and how much they do not want to do the chore segment $[a, b]$ for person i. This is basically a function for how much someone values a chore, and it is different for every person. Some basic restrictions and properties with this function is that d_i is:

- **Non-negative:** $d_i([a, b]) \geq 0$,
- **Additive:** $d_i([a, c]) = d_i([a, b]) + d_i([b, c])$ for $a < b < c$,
- **Normalized:** $d_i([0, 1]) = 1$.

These three properties are pretty simple. The first one states that the function can not be negative, which roughly translates to "a person can never want to do a chore." The second property states that the value of two functions added together whose input is a certain range, when that range is put into it's own function, they will be equal. This means that 2 sets of chores split up is just as annoying to do as 2 sets of chores together. The third property states that when you put in all of the chores into the function, it will always be equal to a constant(in this case, the constant is one), no matter who the function pertains to.

The goal is to divide $[0, 1]$ into n disjoint intervals such that each person i receives a set of chores and the division satisfies a fairness criterion adapted to minimizing the workload on everyone

6.2. **Fairness Criteria for Chores.** The fairness criteria introduced earlier apply, but their interpretations are slightly different:

- **Proportionality:** Each person receives a portion they consider to be no more than $\frac{1}{n}$ of the total chores.
- **Envy-freeness:** No person prefers another person's chore bundle (i.e., no one believes another got off easier).

These goals are often harder to achieve for chores, especially envy-freeness.

6.3. **Algorithmic Approaches for a Solution.** Several algorithms have been adapted or developed for chore division. Some of the more notable among them are as follow:

Cake-Cutting: Some of the procedures mirror the cake-cutting problem, such as a "moving knife" sweeping from right to left, where people call "stop" when the portion swept so far equals their acceptable chore load. These procedures can guarantee proportionality but not envy-freeness as discussed earlier.

Reverse Adjusted Winner: The Adjusted Winner procedure, originally for goods, can be adapted to chores by using the valuations. Items are temporarily assigned and then transferred between people to equalize burdens, often using fractional allocations.

This procedure is usually conducted with two people. Each person gets a certain amount of points, and give points to the chores. In the case below, we have given Bob and Joe 4 chores to do, and 100 points to show us how much they do not like doing each chore. Bob hates getting his hands wet so he does not like dealing with water, and Joe hates being around dust, so they both give a lot of points to those chores.

Task	Bob's point assignment	Joe's point assignment
Washing the dishes	40	25
Taking out the trash	20	25
Vacuuming the house	15	35
Washing the windows	25	15

So now, the person who gave the least amount of points to each chores gets that chore temporarily.

Task	Whose Chore?
Washing the dishes	Joe
Taking out the trash	Bob
Vacuuming the house	Bob
Washing the windows	Joe

The chores that Joe were assigned has a point total of 40, while the point total for Bob is 35. Since Bob has a lower point total, Bob has less chores to do. Now we have to equalize the burden, so we first calculate the ratio of the points that Bob gave per chore by the points that Joe gave for that chore.

Task	Bob's point assignment	Joe's point assignment	Point ratio
Washing the dishes	40	25	1.6
Taking out the trash	20	25	0.8
Vacuuming the house	15	35	0.43
Washing the windows	25	15	1.67

Now, to remove some of the burden from Joe, we take the chore with the lowest ratio that Joe currently has. With this case, the chore that will be partly transferred over is washing the dishes. The amount of chore removed from Joe will go to Bob, and we can calculate the amount using some simple algebra. This is done by setting the point totals equal. The amount of chore will be represented by x.

$$25 + 15 - 25x = 20 + 15 + 40x \tag{6.1}$$

$$x = 0.077 \tag{6.2}$$

After all of that, we get that Bob will do 7.7% of the dishes, take out the trash, and vacuum the house, but Joe with do 92.3% of the dishes and wash the windows. Their point totals are equal at roughly 38.08. This means that they both believe that they have a fair share of the chores and can do them.

Divide and Choose: For two people, an envy-free division of chores can be achieved by one person dividing the chore into two parts of equal burden (in their own view), and the other choosing the part they prefer (or rather, dislike least). This is the exact same as the method discussed in the introduction.

Approximation Algorithms: For more than 2 people, exact envy-free chore division may be impossible due to indivisibility or complexity. Algorithms using discrete approximations divide the chore into many small parts and reassemble bundles to approximate fairness within some margin of error.

6.4. Real-World Applications.

- **Roommate Chores:** Dividing cleaning tasks, taking out trash, doing dishes, etc.
- **Workload Distribution:** Assigning project tasks or time slots that are less desirable.
- **Cost-Sharing:** Dividing a total cost burden among participants fairly (e.g., taxes, rent, group payments).

7. Division of Indivisible Goods

Since indivisible things, like books, homes, or furniture, cannot be divided or shared in smaller quantities, it might be difficult to divide them fairly. Indivisible commodities necessitate discrete allocation procedures that seek to decrease unfairness, which contrasts the earlier forms of division in cake-cutting or chore-division situations, when resources can be divided continuously.

7.1. Modeling an Indivisible Problem.

Let there be a set $G = \{g_1, g_2, \ldots, g_m\}$ of m indivisible items and n people. Each person i has a valuation function $v_i : G \to \mathbb{R}_{\geq 0}$ assigning a non-negative value to each item. The goal is to assign each item to exactly one person such that the allocation meets one or more fairness goals.

7.2. **Definitions for the Division of Indivisible Goods.** For some reason, this area of the study for fair division has a lot more fancy terms.

Definition 7.1. *Envy-Freeness up to One Item*: If one person envies a second person's share, they will not envy the other person's pile after one item is removed from the second person's pile. This is also usually shortened to EF1.

Here is an example of a envy-free division up to one item. Lets say there are two people: Billy and Bob. There are five indivisible items: A ladder, a vacuum, a tennis racket, a guitar, and a suitcase.

Billy's valuations of the items are as follows:

Item	**Value to Billy**
Ladder	6
Vacuum	5
Tennis Racket	4
Guitar	8
Suitcase	3

We can divide these five items in a way so that

- Billy receives: $\{\text{Guitar, Suitcase}\} \to \text{value} = 8 + 3 = 11$
- Bob receives: $\{\text{Ladder, Vacuum, Tennis Racket}\} \to \text{value} = 6 + 5 + 4 = 15$

Billy compares their bundle to Bob's:

$$\text{Bob's value} = 15 \quad \text{vs.} \quad \text{Billy's value} = 11$$

So Billy envies Bob.

However, if we remove one item from Bob's bundle (which has the Ladder, Vacuum, and the Tennis Racket):

$$\text{Bob's bundle without the Ladder} = \{\text{Vacuum, Tennis Racket}\}$$

This means that the value of Bob's share without the ladder $\to$ value $= 5 + 4 = 9$ Then:

$$11 > 9 \Rightarrow \text{Billy no longer envies Bob}$$

Since removing one item (The ladder) from Bob's bundle eliminates Billy's envy, the allocation is envy-free up to one item from Billy's perspective.

Definition 7.2. *Proportionality up to One Item*: Each person receives at least $1/n$ of the total value, up to the value of one item. This is commonly shortened to PROP 1.

To help understand proportionality up to one item, here is an example. We have 3 people: Billy, Bob, and Joe. There are 4 indivisible items: A, B, C, and D. Billy's valuations for the items are as follows:

Item	**Value to Billy**
A	19
B	13
C	14
D	8
Total	54

Billy's proportional share (based on their total value of 51) is:

$$\frac{54}{3} = 18$$

We can give each person the following items:

- Billy receives: $\{C\} \rightarrow$ value = 14
- Bob receives: $\{A\} \rightarrow$ value = 19
- Joe receives: $\{B, D\} \rightarrow$ value = 21

Billy received less than their fair share ($14 < 18$), but we now check if adding one item would make up the difference.

- Add A (from Bob): $14 + 19 = 33 \geq 18$
- Add B (from Joe): $14 + 13 = 27 \geq 18$
- Add D (from Joe): $14 + 8 = 22 \geq 18$

In all cases, adding a single item brings Billy to or above their proportional share. Even though Billy initially received less than one-third of the total value, there exists an item (A, B, or D) that could be added to their bundle to reach or exceed their fair share. For a division to be proportional up to one item, we do this check for everyone else.

Definition 7.3. *Maximin Share Guarantee*: Each person receives at least as much as they could guarantee by dividing the items into n bundles and receiving the least valuable one. This is usually shortened to MMS.

As we did withe the last two ways of division, here is an example. There are two people: Billy and Bob. There are 3 indivisible items: A, B, and C.

Billy's valuations for the items are:

Item	**Billy's Valuation**
A	8
B	5
C	7
Total	20

Billy must divide the items into 2 bundles, assuming that they will receive the less valuable one. We consider all possible partitions:

- Split 1: $\{A, B\}$ and $\{C\}$
  $\Rightarrow$ Values: $8 + 5 = 13$ and $7 \rightarrow$ minimum = 7
- Split 2: $\{A, C\}$ and $\{B\}$
  $\Rightarrow$ Values: $8 + 7 = 15$ and $5 \rightarrow$ minimum = 5
- Split 3: $\{B, C\}$ and $\{A\}$
  $\Rightarrow$ Values: $5 + 7 = 12$ and $8 \rightarrow$ minimum = 8

The best worst-case outcome for Billy is: MMS = 8

Suppose the items are allocated as:

- Billy receives: $\{B, C\} \Rightarrow$ value $= 5 + 7 = 12$
- Bob receives: $\{A\} \Rightarrow$ value = 8

Billy's value is greater than their MMS, so Billy received at least their Maximin Share value. Basically, the split needs to be better than the maximum minimum value, hence the name Maximin.

7.3. **Methods for Dividing Indivisible Things.** Many algorithms exist to divide indivisible goods while satisfying the relaxed fairness criteria mentioned such are envy-free up to one item or proportional up to one item.:

Round-Robin Allocation: People take turns picking their most preferred item in a round-robin order. This simple and intuitive approach ensures high satisfaction when preferences are diverse. It does not guarantee an envy-free up to one item division, but performs well in practice when there are a decent amount of items.

Envy-Free Matching and Graph-Based Approaches This approach models people and items as nodes in a bipartite graph, where:

- One set of nodes represents people, and the other set represents items.
- Edges connect each person to each item and are weighted based on individual preferences or utility.
- A matching corresponds to an allocation of items to people.

Algorithms such as the *Hungarian algorithm* are used to find a maximum-weight matching, maximizing total satisfaction.

Although these algorithms primarily aim for efficiency, they can be adapted to approximate fairness notions such as envy-freeness up to one item. Down below, you can see an example of the graph based approach.

Lets say we have 3 people, Billy, Bob, and Joe, and 3 items labeled A, B, and C. In the table below, you can see how much each person values each item.

	Billy	Bob	Joe
A	9	6	3
B	4	8	7
C	5	2	10

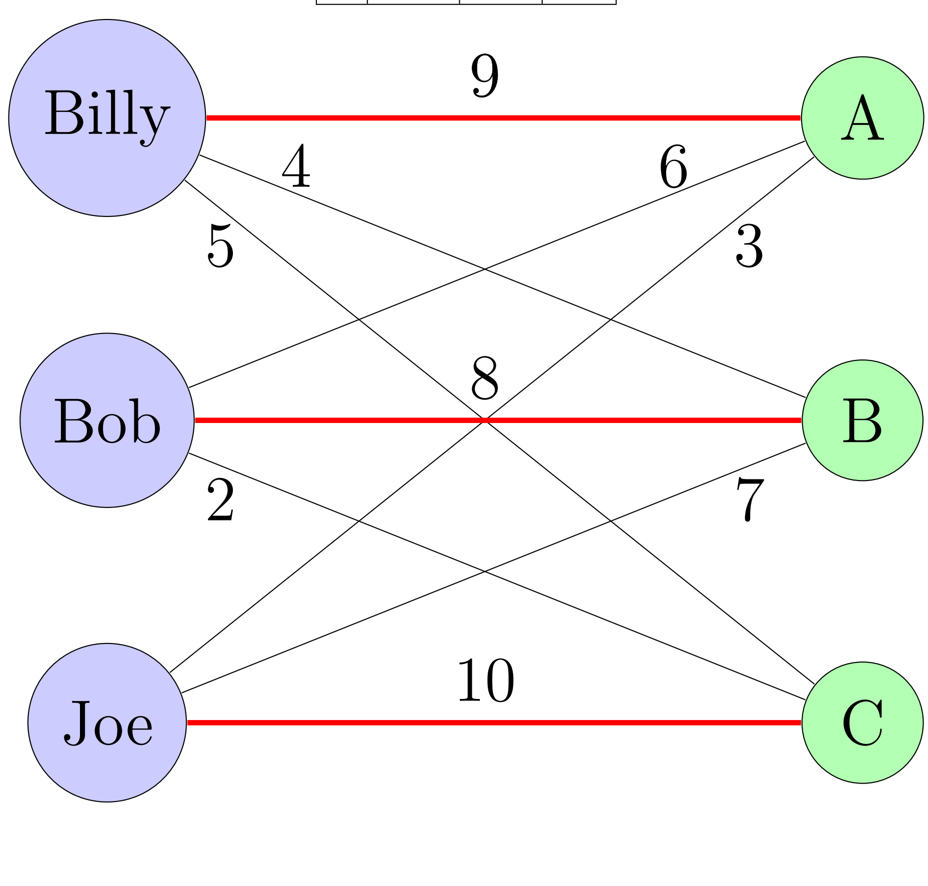

Figure 6

- A is assigned to Billy (Value = 9)
- B is assigned to Bob (Value = 8)
- C is assigned to Joe (Value = 10)
- Total value: $9 + 8 + 10 = 27$

- No person envies another's assignment.
- Each person either prefers their own item more or values others' less.
- The result is envy-free, and thus satisfies EF1.

Greedy Algorithms for Envy-Freeness Up to One Item

From our adjusted criteria for envy-freeness, one thing that is guaranteed to happen is for a envy-free up to one item division to always exist for additive valuations.

One of the many procedures is the *Lipton et al. protocol*, which follows the steps below:

(1) Begin with any arbitrary allocation of items.
(2) While someone envies another by more than one item, transfer an item from the envied person to the envious person.
(3) Repeat this process until no such envy exists.

This algorithm is guaranteed to terminate in a finite number of steps, resulting in an EF1 allocation. Here is a run-through of it in action. With the people Billy and Bob, and the items A, B, and C.

Valuations:

	A	B	C
Billy	6	4	1
Bob	3	5	2

Let:

- Billy get B and C
- Bob get A

Valuations:

- Billy: $v_i(B + C) = 4 + 1 = 5$
- Bob: $v_o(A) = 3$

Envy:

- Bob envies Billy: $v_o(B + C) = 5 + 2 = 7 > 3$
- Envy is not EF1: even if Bob removes one item (say B), $v_o(C) = 2$, and $3 < 2 \Rightarrow$ still envies

Transfer B from Billy to Bob.
New allocation:

- Billy: $\{C\}$
- Bob: $\{A, B\}$

Valuations:

- Billy: $v_i(C) = 1$
- Bob: $v_o(A + B) = 3 + 5 = 8$

Now Billy envies Bob: $v_i(A + B) = 6 + 4 = 10 > 1$
But EF1 check: If we remove item A or B, Billy's envy reduces:

- $v_i(B) = 4 \Rightarrow 4 > 1$ (still envy)

- $v_i(A) = 6 \Rightarrow 6 > 1$ (still envy)

So it's not EF1 yet. Transfer one more item.
New allocation:

- Billy: $\{A, C\}$
- Bob: $\{B\}$

Valuations:

- Billy: $v_i(A + C) = 6 + 1 = 7$
- Bob: $v_o(B) = 5$

Bob envies Billy: $v_o(A + C) = 3 + 2 = 5 < 5 \Rightarrow$ no envy
Billy does not envy Bob either.
Maximin Share Approximations
Remember the Maximin Share? One thing to know is that a Maximin Share allocation may not exist when items are indivisible. To address this, approximation algorithms are used to ensure that each player receives at least a fraction of their MMS value.

Another result is the 3/4-MMS guarantee, which ensures that each player receives at least 75% of their maximin share. These approximations strike a balance between fairness and computational feasibility.

Concept	**Goal**	**Guarantee Type**
Envy-Free Matching	Maximize total satisfaction using preference-weighted bipartite graphs	Approximate EF1
Greedy EF1 Algorithms	Eliminate strong envy through iterative item transfers	Always EF1
MMS Approximations	Guarantee each player a fair fraction of their worst-case fair share	Typically 3/4-MMS

Rounding from Continuous Models Another powerful idea is to start with a fractional division (e.g., using a market equilibrium approach or a linear program), then apply rounding procedures to convert fractional allocations into discrete ones, while maintaining approximate fairness and efficiency.

7.4. **Applications.**

- **Inheritance Division:** Distributing family heirlooms among siblings.
- **Classroom Resources:** Assigning tablets or books to students.
- **Housing Allocation:** Assigning dorm rooms or apartments to students or tenants.
- **Work Tasks:** Assigning project roles or responsibilities that cannot be split.

8. Acknowledgments

I want to thank my mentor, Faizan Hussaini for the valuable feedback and tips throughout the process of writing this paper. I would like to also thank Simon Rubenstein-Salzedo for organizing Euler Circle.

References

[Aus82] Alfred K. Austin. Sharing a cake. *Mathematical Gazette*, 66(437):212–215, 1982.

[BP15] Simina Branzei and Ariel D. Procaccia. Discrete cake cutting: Complexity and fairness. In *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence*, pages 568–574. AAAI Press, 2015.

[BT96] Steven J. Brams and Alan D. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution.* Cambridge University Press, 1996.

[BT00] Steven J. Brams and Alan D. Taylor. The adjusted winner procedure: An experiment in negotiation. *Journal of Conflict Resolution*, 44(1):48–69, 2000.

[Bud11] Eric Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6):1061–1103, 2011.

[LMMS04] Richard J. Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. On approximately fair allocations of indivisible goods. In *Proceedings of the 5th ACM Conference on Electronic Commerce (EC)*, pages 125–131. ACM, 2004.

[Pro16] Ariel D. Procaccia. Cake cutting algorithms. In Piotr Faliszewski, Edith Hemaspaandra, Lane A. Hemaspaandra, and Jörg Rothe, editors, *Handbook of Computational Social Choice*, chapter 13, pages 311–329. Cambridge University Press, 2016.

[RW98] Jack Robertson and William Webb. *Cake-Cutting Algorithms: Be Fair If You Can.* AK Peters, 1998.

[Su99] Francis Edward Su. Rental harmony: Sperner's lemma in fair division. *The American Mathematical Monthly*, 106(10):930–942, 1999.