The Probabilistic Method in Combinatorics

Simon Meyers
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Mentor: Simon Rubinstein-Salzedo

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What Is the Probabilistic Method?

- A non-constructive technique: proves that certain objects exist **without constructing them**.
- Idea: Define a probability space over possible structures. If the probability that a "bad" event happens is < 1, then a "good" object must exist.
- Pioneered by Paul Erdős; now central to modern combinatorics.
- Often used to show existence of graphs, sets, or colorings with desired properties.

Philosophy of the Probabilistic Method

Key Principle

If you randomly select an object and the expected number of "bad" events is less than 1, then there exists an object with *no* bad events.

- **Non-constructive**: We don't find the object, just prove it exists.
- **Applications**: Lower bounds, existence proofs, sometimes even algorithmic versions.
- A shift in mathematical thinking: Using randomness to prove certainty.

The First Moment Method

Theorem (Expectation Method)

Let X be a non-negative integer-valued random variable. If $\mathbb{E}[X] < 1$, then $\mathbb{P}(X = 0) > 0$.

- We show that a "bad" configuration happens with probability less than 1.
- Then, with positive probability, no bad event occurs.
- Proves existence non-constructively.

Key Idea of the First Moment Method

- Define a random variable X that counts the number of undesirable (bad) objects — e.g., monochromatic cliques.
- Compute $\mathbb{E}[X]$.
- If $\mathbb{E}[X] < 1$, then with positive probability, $X = 0$.
- Therefore, a structure avoiding all bad configurations must exist.

This is the method Erdős used in his groundbreaking 1947 paper to show that $R(k, k) > 2^{k/2}$.

Ramsey Numbers

Definition: The Ramsey number $R(k,k)$ is the smallest n such that every red-blue coloring of the edges of K_n contains a monochromatic K_k.

Historical Insight: Ramsey theory asks: how large must a system be before structure becomes inevitable?

Known Bounds (as of 2020):

$$2^{k/2} < R(k,k) \leq 4^k$$

The lower bound is from Erdős's probabilistic method; the upper bound is from constructive combinatorics.

Open Problem: The gap between these bounds remains one of the biggest challenges in extremal combinatorics.

Probabilistic methods give the best known lower bounds.

Erdős's Lower Bound on Ramsey Numbers

Goal

Prove that $R(k,k) > n$ for some large n.

- Consider a random 2-coloring of edges of K_n.
- Let X be the number of monochromatic K_k subgraphs.
- Each K_k is monochromatic with probability $2 \cdot 2^{-\binom{k}{2}}$.
- So:

$$\mathbb{E}[X] = \binom{n}{k} \cdot 2 \cdot 2^{-\binom{k}{2}}$$

- If $\mathbb{E}[X] < 1$, then there exists a 2-coloring with no monochromatic K_k.

The Second Moment Method

Motivation: First moment shows $\mathbb{E}[X] > 0$, but maybe $X = 0$ still occurs frequently.

Idea: Use variance to show $X > 0$ with positive probability.

Chebyshev's Inequality: If X is a random variable with finite variance, then:

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq \frac{\text{Var}(X)}{t^2}$$

Application: Counting Hamiltonian cycles in $G(n, p)$ or estimating size of longest common subsequences.

Conclusion: A stronger tool when the first moment alone is not enough.

Alterations

Key Idea: Start with a random object and then modify it to eliminate flaws.

Origin: Erdős and Rényi pioneered this method.

Process:

1. Construct a random object (e.g., graph).
2. Identify and remove elements causing bad events.

Example: Build a graph with many edges but no large independent set or clique by deleting problematic vertices.

Benefit: Often gives better bounds than the First Moment Method alone.

The Lovász Local Lemma (LLL)

Problem: What if bad events are not fully independent?

Symmetric Form: If each bad event has probability p and depends on at most d others, and:

$$ep(d+1) \leq 1$$

then with positive probability, none of the bad events occur.

Applications:

- Hypergraph colorings
- CNF satisfiability (e.g. k-SAT)
- Constructing Ramsey graphs

Constructive Version: Moser–Tardos Algorithm uses resampling to find a good configuration.

Constructive Algorithms

Key Question: Can we actually find the objects that probabilistic proofs show exist?

Answer: Yes — in many cases via algorithmic derandomization.

Example: Moser-Tardos Algorithm

- Applies to the Lovász Local Lemma
- Uses resampling to eliminate bad events
- Converges efficiently under dependency constraints

Philosophical Shift: Randomness not just a tool for existence, but a guide to construction.

Other Applications (1): Graph Theory

Turán-type Extremal Problems:

- How dense can a graph be without containing a fixed subgraph?
- Probabilistic constructions give strong lower bounds.

Chromatic Number Bounds:

- Show that graphs with large girth and large chromatic number exist.
- Random constructions achieve this.

Other Applications (2): Number Theory Geometry

Number Theory: Sum-Free Sets

- Subsets with no 3-term arithmetic progressions.
- Random selection shows large sum-free sets exist.

Geometry: Discrepancy and General Position

- Discrepancy theory: Balance subsets in partitions.
- Construct point sets in general position.

Other Applications (3): Algorithms

Approximation Algorithms:

- Use randomized rounding in LP relaxations.
- Examples: Max-Cut, Set Cover.

Derandomization:

- Use method of conditional expectations.
- Boosts theoretical guarantees for algorithms.

Takeaway: Probabilistic method is not only existential — it influences algorithm design.

Thank You!

Questions?

- Thank you for your time!
- I'm happy to take any questions.

Presented by: Simon Meyers