The Black–Scholes Model

Sankar Jonnalagadda

July 14, 2025

Historical Context and Motivation

- 1973: Fischer Black, Myron Scholes, and Robert Merton
- Foundation for modern financial engineering

Stochastic Processes

A stochastic process is a collection of random variables indexed by time:

- Denoted $\{X_t\}_{t \in T}$, where each X_t maps outcomes to values
- Examples:
  - Stock price S_t over time
  - Brownian motion W_t as building block
- Key properties:
  - Independent increments of $X_{t+h} - X_t$

Brownian Motion: Definition and Properties

A real-valued stochastic process $\{W_t\}_{t\geq 0}$ such that:

- $W_0 = 0$ (starts at zero)
- Independent increments: $W_t - W_s$ independent for $t > s$
- Continuous path

Key consequences:

- $E[W_t] = 0$, $\mathrm{Var}(W_t) = t$
- Paths are nowhere differentiable (fractal)
- Quadratic variation over $[0, t]$ equals t

Geometric Brownian Motion

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t.$$

- μ: drift parameter
- σ: volatility parameter
- Interprets real-world stock movement

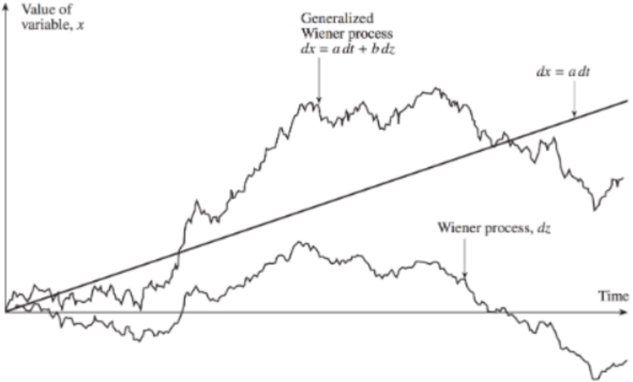
Geometric Brownian Motion: Risk-Neutral Perspective

In a risk-neutral world, all assets are expected to grow at the risk-free rate r. Thus, we replace the real-world drift μ with r in the GBM:

$$\boxed{dS_t = r\, S_t\, dt \;+\; \sigma\, S_t\, dW_t.}$$

- **Drift term** $r\, S_t\, dt$: the "fair" expected increase per unit time
- **Diffusion term** $\sigma\, S_t\, dW_t$: random fluctuations around the drift
- **Implication for pricing:** Option values depend only on r and σ, not on investors' risk preferences.

Geometric Brownian Motion Illustration



Itô Process Form

An Itô process satisfies:

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, dW_t.$$

- General framework for SDEs
- Functions a, b are adapted processes

Itô's Lemma

For twice-differentiable $f(x, t)$ and Itô process X_t:

$$d(f(X_t)) = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)\, (dX_t)^2$$

- Chain-rule for randomness

Quadratic Variation

In stochastic calculus, Brownian increments scale as $O(\sqrt{\Delta t})$, so their squares scale as $O(\Delta t)$:

$$(W_{t+\Delta t} - W_t)^2 = O(\Delta t).$$

The limit of the sum of these squared increments defines the quadratic variation:

$$[W]_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(W_{t_{i+1}} - W_{t_i}\right)^2 = t.$$

Properties due to Quadratic Variation:

- $(dW_t)^2 = dt$
- $dW_t\, dt = 0$
- $dt^2 = 0$

Key Assumptions

- Risk-free rate r constant; no dividends or carrying costs
- Frictionless markets: no transaction costs; unlimited borrowing/lending
- Short-selling of the underlying asset is permitted
- Underlying asset follows Geometric Brownian Motion:

$$dS_t = \mu\, S_t\, dt \; + \; \sigma\, S_t\, dW_t$$

Deriving the Black-Scholes PDE

Our option price will be denoted as $V(S_t, t)$ as the option price is dependent on time and stock price.

Apply Itô's Lemma to $V(S_t, t)$ where $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$:

$$dV = \frac{\partial V}{\partial t}\, dt + \frac{\partial V}{\partial S}\, dS_t + \tfrac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS_t)^2$$

Expanded Differential of V

Substituting $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$ and $(dS_t)^2 = \sigma^2 S_t^2\,dt$:

$$dV = \left(\frac{\partial V}{\partial t} + \mu\,S_t\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S^2}\right)dt \;+\; \sigma\,S_t\frac{\partial V}{\partial S}\,dW_t$$

Delta-Hedged Portfolio Construction

Create a portfolio of one call option, V, and Δ shares of stock:

$$\Pi = V + \Delta\, S_t.$$

Then,

$$d\Pi = (\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \Delta \mu S_t)dt + \sigma S_t(\frac{\partial V}{\partial S} + \Delta)dW_t$$

Eliminating the Stochastic Term

The dW_t is the source of randomness. To eliminate this term, we can set $\Delta = -\frac{\partial V}{\partial S}$

Our equation now becomes:

$$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2})dt$$

No-Arbitrage

Since we assume there is no arbitrage, and there is no random term, the portfolio grows at a risk-free rate r, such that

$$d\Pi = r\Pi dt = r(V - S_t \frac{\partial V}{\partial S})dt$$

Black Scholes PDE

Using the substitution from the pervious slide, we can equate these terms to get:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} = r(V - S_t \frac{\partial V}{\partial S})$$

Rearranging the terms, we get the Black-Scholes PDE

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} - rV = 0$$

Black–Scholes Pricing Formula

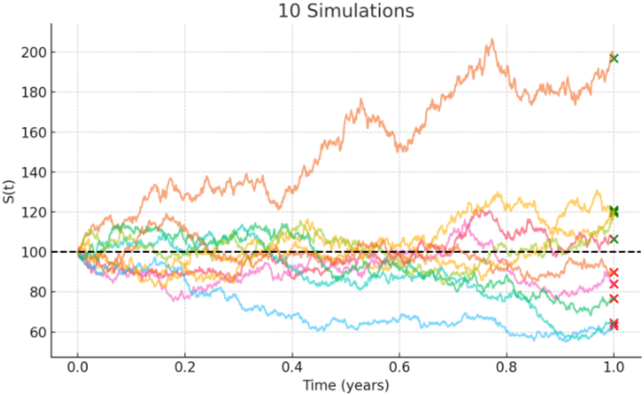
$$C = N(d_1)\, S_t - N(d_2)\, Ke^{-rt} \quad \text{where}$$

$$d_1 = \frac{\ln \frac{S_t}{K} + \left(r + \frac{1}{2}\sigma^2\right) t}{\sigma\sqrt{t}}, \quad d_2 = d_1 - \sigma\sqrt{t}.$$

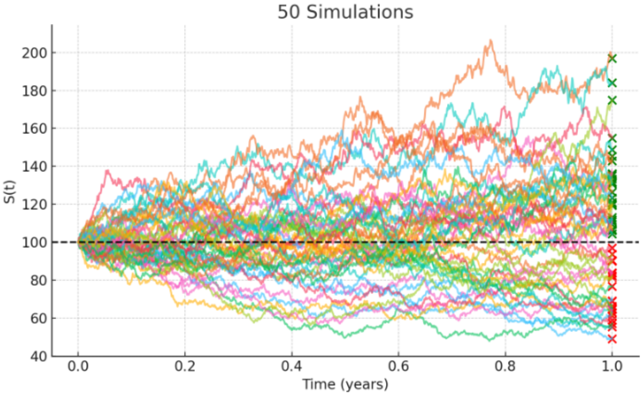
Notation:

- C: call option price
- $N(x)$: cumulative normal distribution
- S_t: spot price of underlying
- K: strike price
- r: risk-free interest rate
- t: time to maturity
- σ: asset volatility

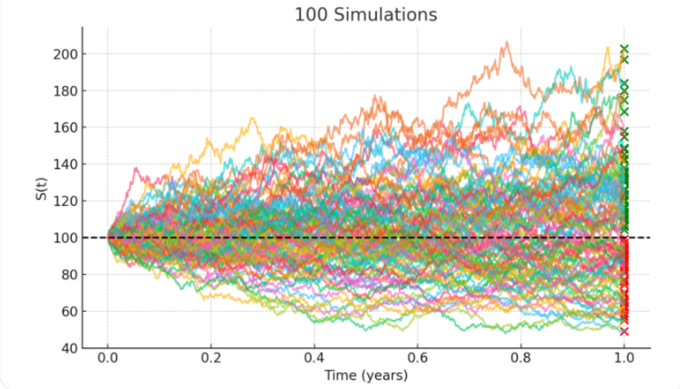
Applications



Applications



Applications



Applications

These simulations can help us estimate the price of a call option; If we assume $S_0 = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.3$, $T = 1$, then the Monte Carlo estimation gives us

$$\hat{C} \;=\; e^{-rT}\,\frac{1}{N}\sum_{i=1}^{N}\max\bigl(S_T^{(i)} - K,\,0\bigr).$$

1. 10 simulations: Option price estimate of 5.91 dollars
2. 50 simulations: Option price estimate of 16.73 dollars
3. 100 simulations: Option price estimate of 12.53 dollars

Black-Scholes pricing equation tells us the theoretical value of a call option. In this case, the true value is 14.23 dollars.

References

- Øksendal, B. (2003). *Stochastic Differential Equations*, University of California, Los Angeles
- Dunbar, S. "Deriving the Black–Scholes Solution," University of Nebraska–Lincoln.
- "Lecture 7: Black–Scholes Theory," Department of Statistics, University of Chicago.
- Haugh, M. (2015). *Foundations of Financial Engineering*, Columbia University.
- Björk, T. (2009). *Arbitrage Theory in Continuous Time*.