BLACK-SCHOLES MODEL

SANKAR JONNALAGADDA

Abstract. This paper introduces fundamental concepts from probability theory, Brownian motion, and stochastic calculus to show the basic ideas behind dynamic pricing models. From these ideas, the paper will go on to derive the Black-Scholes partial differential equation (PDE). By solving this PDE, the paper derives the Black-Scholes pricing model, an expression for the value of a European call option. Finally, the applications of the model are explored through the simulation of stock prices.

Date: June 2025.

1. Introduction

The Black-Scholes-Merton model, or more commonly known as the Black-Scholes model, is a mathematical model that shows the theoretical estimate of the price of a European call option. This paper will serve as an introduction to the model: We will explore fundamental concepts in Brownian motion, stochastic calculus, probability theory (Sections 2.1, 2.2, 2.3) and derive the following Black-Scholes pricing equation with Ito's Lemma (Section 4)

$$C = S_0 N(d_1) - N(d_2) K e^{-rt}$$

$$d_1 = \frac{\ln \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right) t}{\sigma \sqrt{t}}$$

$$d_2 = d_1 - \sigma \sqrt{t}$$

Where C is the fair value of a European call option, S_0 is the initial price of a stock, K is the strike price, $N(x)$ is the cumulative normal distribution (CDF) of a standard normal variable, σ is the volatility of the asset, r is the risk-free interest rate, and t is the time until maturity. Additionally, after deriving this equation, we will look into the applications of the model, simulating stock prices using two different methods (Section 5.1, 5.2). By the end of this paper, the reader should have both an analytical and conceptual understanding of the mathematical ideas within the Black-Scholes framework.

2. Mathematical Background Knowledge

2.1. **Probability.** Probability is the basis for modeling uncertainty. As a result, it is imperative that we understand basic probability theory before we move on to more advanced topics. However, probability theory is a vast subject, and this paper will only include concepts that are important for the discussion of the mathematical framework behind the Black-Scholes model. Specifically, this Section (2.1) will only introduce concepts such as probability spaces, random variables, probability density functions (PDF), and the cumulative distribution function (CDF). Readers seeking a more complete understanding of probability theory are encouraged to reference *Probability and Random Processes* by Geoffrey Grimmett and David Stirzaker.

Definition 2.1. *A probability space* is a mathematical construct consisting of three elements: $(\Omega, \mathcal{F}, \mathbb{P})$.

Ω: The sample space, which is the set of all possible outcomes in an experiment.

$\mathcal{F}$: A σ-*algebra* of Ω, which is a collection of subsets under Ω with three properties.

(1) $\emptyset \in \mathcal{F}$
(2) If an event $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
(3) If $A_1, A_2, \cdots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

In finance, a σ-algebra can be thought of as an information set: It specifies what events are known, and helps figure out the probability associated with those outcomes.

$\mathbb{P}$: *A function* that assigns a probability to each event in $\mathcal{F}$.

For example, a six-sided die has the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$. Now, the event that an odd number is rolled is A={1,3,5}.

Since $A = \{1, 3, 5\}$, the probability function $\mathbb{P}$ shows that

$$\mathbb{P}(A) = \mathbb{P}(\{1, 3, 5\}) = \frac{1}{2}$$

Definition 2.2. *A random variable* X is a function that assigns a real number to each outcome in the sample space Ω. There are two types of random variables: Discrete and continuous. A discrete random variable has values that are finite and countable over some interval. A continuous random variable takes on some uncountable number of values over an interval.

Definition 2.3. *A cumulative distribution function (CDF)*, denoted as $F_X(x)$, shows the probability that the discrete random variable X is less than or equal to x. This can be represented by $F_X(x) = \mathbb{P}(X \leq x)$

Definition 2.4. *A probability distribution function (PDF)*, denoted as $f_X(x)$, shows how likely it is for the continuous random variable X to fall within a small interval around x. This can be represented by

$$f_X(x) = \lim_{\Delta \to 0^+} \frac{\mathbb{P}(x < X \leq x + \Delta)}{\Delta} = \frac{dF_X(x)}{dx} = F'_X(x)$$

Definition 2.5. *Expected Value*, denoted as $\mathbb{E}[X]$, is the weighted average of the outcomes and probabilities (Also known as the mean) of a random variable X. For example, the expected value for a six-sided die is:

$$\mathbb{E}[X] = \sum_{i=1}^{6} i \cdot \frac{1}{6} = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = \frac{21}{6} = 3.5$$

Definition 2.6. *Variance*, denoted as $Var(X)$, is a measure of how spread the distribution of a random variable is. This concept is quite important for us, as variance can be used to see how much risk an investment carries. Variance is represented by $Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

With our understanding of basic probability theory from definitions 2.1-2.6, we are now ready to move onto our next topic.

2.2. **Brownian Motion.**

Brownian motion is essential for our study of random movements. It's key properties of independently distributed increments, zero starting value, and continuous paths allow it to be a good choice for modeling asset price fluctuations. This section introduces these concepts to help the reader understand this critical component of the Black-Scholes model.

Definition 2.6. A *stochastic process* is a collection of random variables $\{X_t\}_{t\in T}$ that describe how a quantity changes over time in a random way. Each X_t represents the value of the process at time t, and the set T is usually a set of time points (such as $[0, \infty)$).

Definition 2.7.*Brownian motion* (*The Wiener process*) is a real-valued stochastic process $\{W(t)\}_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying:

1. $W(0) = 0$.
2. For any $0 \leq t_1 < t_2 < \cdots < t_n$, the increments $W(t_i) - W(t_{i-1})$ are independent.
3. For all $s < t$, $W(t) - W(s) \sim \mathcal{N}(0, t-s)$.
4. The map $t \mapsto W(t)$ is almost surely continuous.

These properties show us that standard Brownian motion randomly fluctuates around zero such that at any time t, $\mathbb{E}[W(t)] = 0$. However, real-world stock prices do not fluctuate around zero. Rather, they always stay positive. To account for this, we add a positive drift term. The drift shifts the overall behavior upward, fluctuating the Brownian motion over some function.

Definition 2.8. *Geometric Brownian Motion* is a stochastic process in which the logarithm of a random quantity follows Brownian Motion. In other words, Geometric Brownian Motion is a specific exponential transformation of Brownian Motion.

Definition 2.9 *Stochastic Differential Equation*, also known as an SDE, is a differential equation where one of the terms is a stochastic process. SDEs are the basis of modeling for growth behavior, jump processes, and more importantly for us: Stock prices.

Now, we can introduce a basic model of our asset (S_t), defined by: $dS_t = \mu S_t dt + \sigma S_t dW_t$

This SDE will be essential as we move on, as it's used in our later derivations for the Black-Scholes model. Additionally, to help the reader visualize this section, the following graphic has been included.

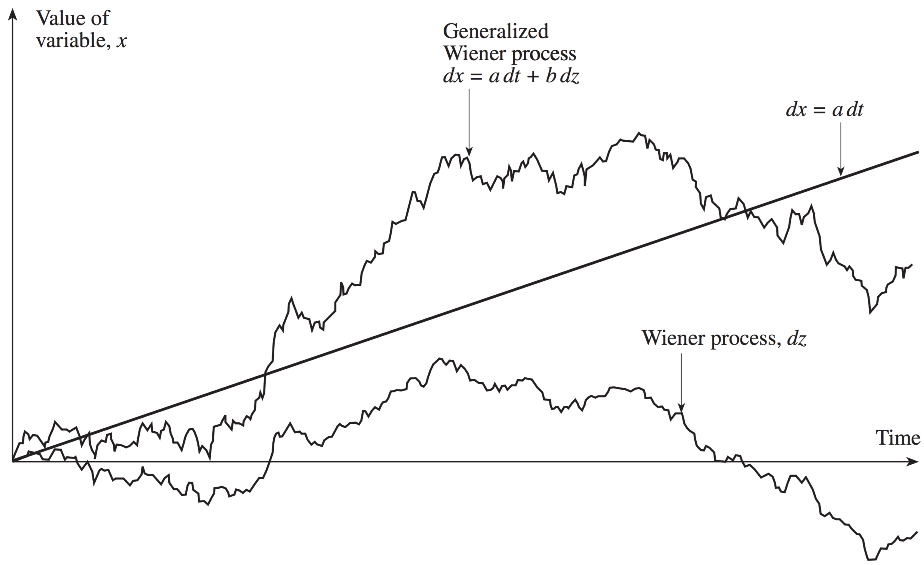


Figure 1

2.3. **Stochastic Calculus.**

Stochastic calculus finds the rate of change of a stochastic model, which is essential for predicting future asset prices. In this section we introduce Itô processes, Itô's lemma, and stochastic differential equations.

Definition 3. An *Itô integral* of a process X_t with respect to a Brownian motion W_t over $[0, T]$ is defined as the mean-square limit of Riemann sums

$$\int_0^T X_t \, dW_t \ := \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} X_{t_i} \left(W_{t_{i+1}} - W_{t_i} \right),$$

Conceptually, an Ito integral adds extremely small increments of dW_t over $[0, T]$, each weighted by the stochastic process X_t; In finance, this integral gives you the cumulative gain or loss generated by continuously adjusting your position X_t in response to random changes in dW_t

Definition 3. An *Itô process* on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a process X_t of the form

$$X_t = X_0 + \int_0^t \mu(s, X_s) \, ds \ + \int_0^t \sigma(s, X_s) \, dW_s,$$

which can also be written in the form

$$dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t$$

where μ and σ are adapted processes and W_t is a standard Wiener process.

Ito processes are important because they are the generalized form of different asset models. In addition, they will be used to solve pricing partial differential equations, form risk-neutral measures, and appear in Ito's Lemma.

Theorem 3.1: Itô's Lemma. Let X_t be an Itô process

$$dX_t = \mu_t \, dt + \sigma \, dW_t,$$

and let $f(x)$ be a twice differential function that is continuous. Then, $f(X_t)$ is again an Itô process, and

$$d(f(X_t)) = f'(X_t) \, dX_t + \tfrac{1}{2} f''(X_t) \, (dX_t)^2$$

Theorem 3.2: Multiplication Properties of Stochastic Calculus.

$$\begin{aligned} (dW_t)^2 &= dt, \\ dW_t \, dt &= 0, \\ dt^2 &= 0. \end{aligned}$$

These properties are extremely useful in our derivation of the Black-Scholes model, and will be used in multiple examples throughout this paper. A quick proof will be shown for each:

For the property $(dW_t)^2 = dt$,

(1) The time interval $[0, T]$ is split into n equal subintervals of length $\Delta t = T/n$
(2) We define the ith subinterval $[t_i, t_{i+1}]$ such that $\Delta W_i = W_{i+1} - W_{t_i}$
(3) Brownian motion tell us that $Var(\Delta W_i) = \Delta t$, so the standard deviation is $\sqrt{Var(\Delta W_i)} = \sqrt{\Delta t}$.
(4) So, $(\Delta W_i)^2$ and Δt are of the same order.
(5) As they are same order, $\Sigma(\Delta W_i)^2 = \Sigma \Delta t$, giving $(dW_t)^2 = dt$

For the property $dW_t dt = 0$

(1) We know that Brownian motion has increments of order $\Delta W = \sqrt{\Delta t}$ and the time increment is $t = \Delta t$
(2) The product $(\Delta W)(\Delta t) = (\sqrt{\Delta t})(\Delta t) = \Delta t^{3/2}$
(3) As the increments of t approach zero, $\Delta t^{3/2}$ becomes negligible compared to Δt.

For the property $dt^2 = 0$

(1) We know that the time increment of t is just the order of Δt, so $(\Delta t)^2 = \Delta t^2$
(2) As $\Delta t \to 0$, Δt^2 approaches zero much faster than Δt
(3) So, Δt^2 is negligible compared to Δt, and we get $(dt)^2 = 0$

If the reader is interested in a more in-depth proof, he or she is recommended to read Chapter 5.1, An Existence and Uniqueness Result from *Stochastic Differential Equations* by Bernt Øksendal.

Example 1: let $f(x) = x^2$ and $X_t = W_t$, where $\mu_t = 0$ and $\sigma = 1$. Then, $dX_t = dW_t$, $f'(x) = 2x$, $f''(x) = 2$

Using Itô's Lemma, we find

$$d\big(f(X_t)\big) = 2X_t\,dX_t + \tfrac{1}{2}\,(2)\,(dX_t)^2.$$

Given $X_t = W_t$,

$$d\big(W_t^2\big) = 2W_t\,dW_t + (dW_t)^2.$$

Simplifying with $(dW_t)^2 = dt$,

$$d\big(W_t^2\big) = 2W_t\,dW_t + dt.$$

This final equation allows us to calculate how a function changes when input into a random process. Specifically for this example, we can calculate how the function $g(x) = x^2$ changes when applied to the stochastic process $X_t = W_t$.

Example 2: Assuming that $f(S_t) = ln(S_t)$, we can find the closed form solution of Geometric Brownian Motion using Ito's Lemma.

Starting with Itô's Lemma, we get

$$df(S_t) = f'(S_t)\,dS_t + \tfrac{1}{2}\,f''(S_t)\,(dS_t)^2.$$

Substituting $f'(S_t) = \frac{1}{S_t}$, $f''(S_t) = -\frac{1}{S_t^2}$ and $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$, we have

$$df(S_t) = \frac{1}{S_t}\big(\mu S_t\,dt + \sigma S_t\,dW_t\big) + \tfrac{1}{2}\Big(-\frac{1}{S_t^2}\Big)\big(\mu S_t\,dt + \sigma S_t\,dW_t\big)^2.$$

Now, simplifying using $(dt)^2 = 0$, $(dW_t)^2 = dt$, we get

$$df(S_t) = \mu\,dt + \sigma\,dW_t - \tfrac{\sigma^2}{2}\,dt = \sigma\,dW_t + \Big(\mu - \tfrac{\sigma^2}{2}\Big)\,dt.$$

Integrating from 0 to t:

$$\ln S_t - \ln S_0 = \int_0^t \sigma\,dW_s + \int_0^t \Big(\mu - \tfrac{\sigma^2}{2}\Big)\,ds = \sigma\,W_t + \Big(\mu - \tfrac{\sigma^2}{2}\Big)t.$$

Solving for S_t gives us

$$S_t = S_0\, e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})\, t}.$$

This expression in fundamental to fields like quantitative finance and risk management because it follows most market observations: It is strictly positive, yields log-normally distributed returns, and is easy to model. Additionally, this formula is used to derive the Black-Scholes pricing formula under a risk-neutral measure.

3. Financial Background Knowledge

The final step before we dive into the Black-Scholes model is that the reader has financial knowledge. Black-Scholes, after all, does indeed model the financial world, and thus it is necessary to understand what is going on conceptually.

Definition 3.3. An *Asset* is anything that is owned and has monetary value. Black-Scholes and Brownian motion are used to model these assets.

One type of asset we have worked with already are stocks. During our study of Geometric Brownian Motion, we modeled a stock as $S_t = \mu S_t dt + \sigma S_t dW_t$. Later, in our study of stochastic calculus, we found the closed-form solution of Geometric Brownian Motion using a log transform of $f(S_t) = ln(S_t)$.

Definition 3.4. A *financial option* is an agreement to buy or sell an asset at a specified future date.

Definition 3.5. A *European Call Option* is a type of financial derivative contract where the individual has the right, but not the obligation, to buy or sell an asset at a predetermined strike price K, at a fixed expiration date T.

Definition 3.6. The *Strike Price* (K) is the price at which you buy the underlying if you exercise your right to buy or sell.

Definition 3.7. The *Premium* is the price you pay to have the right, but not the obligation, to buy or sell an asset.

Example 3: An Apple stock is priced at \$200 at time t_0. An individual pays a premium of \$5 to buy an Apple stock at a strike price of \$200 at some future time T. Now, three situations can occur:

(1) At time T, the Apple stock goes up in value to \$250. The individual exercises the right to buy for \$200 and resells at the market price of \$250. Profit:

$$250 \text{ (Market Value)} - 200 \text{ (Strike Price)} - 5 \text{ (Premium)} = 45.$$

(2) At time T, the Apple stock goes up in value to \$205. The individual exercises the right to buy for \$200 and resells at the market price of \$205. Profit (break even):

$$205 \text{ (Market Value)} - 200 \text{ (Strike Price)} - 5 \text{ (Premium)} = 0.$$

(3) At time T, the Apple stock stays at \$200. The individual exercises the right to buy for \$200 and resells at the market price of \$200. Profit:

$$200 \text{ (Market Value)} - 200 \text{ (Strike Price)} - 5 \text{ (Premium)} = -5.$$

The understanding of a European call option is essential as the point of the Black-Scholes model is to calculate the fair value price of a premium.

Definition 3.8. The *risk-free rate* is the rate of return on an investment with no risk. The risk-free rate is determined by market supply and demand.

Definition 3.9. A *risk neutral* measure means that every asset, on average, earns the risk-free rate. Mathematically, we represent this as

$$C(A,0) \;=\; e^{-rt}\,\mathbb{E}\big[C(A,t)\big]$$

where

$$\begin{aligned} C(A,0) \quad &: \quad \text{The initial price of the contract} \\ e^{-rt} \quad &: \quad \text{The discount factor over period } t \text{ at the risk-free rate } r, \\ \mathbb{E}\big[C(A,t)\big] \quad &: \quad \text{The risk-neutral expected payoff of the contract at time } t. \end{aligned}$$

Definition 4.0 A *Portfolio* is a collection of assets usually held by an investor or trader.

Definition 4.1 *Arbitrage* is a risk-less portfolio that is not the risk-free asset. The Black-Scholes model assumes that there is no arbitrage so that the return of a hedged portfolio must equal the risk-free rate.

4. Derivation with Ito's Lemma

This section will combine everything we have seen so far to derive the Black-Scholes model. This will involve two steps: Deriving the Black-Scholes PDE and solving the PDE to get the equation for a European call option. Finally, before we start, it's important to note some assumptions that help us throughout this process.

Assumptions

(1) The short-term interest rate r is known and constant.
(2) The underlying asset pays no dividends during the option's life.
(3) There are no transaction costs or taxes on trading.
(4) Investors may trade fractional shares and borrow or lend unlimited amounts at the risk-free rate.

(5) Short-selling of the underlying asset is permitted without restriction.
(6) The underlying asset follows Geometric Brownian Motion, represented as

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Deriving the Black Scholes PDE

(1) First, our option price will be denoted as $V(S_t, t)$ as the option price is dependent on time and stock price.
(2) Now, we need to see how the function V evolves over time. Using Ito's Lemma on the function $V(S_t, t)$, we find that

$$dV = \frac{\partial V}{\partial t}\, dt + \frac{\partial V}{\partial S}\, dS_t + \tfrac{1}{2}\frac{\partial^2 V}{\partial S^2}\,(dS_t)^2$$

(3) Using properties from Theorem 3.2, we find the quadratic term

$$(dS_t)^2 = (\mu S_t dt + \sigma S_t dW_t)^2 = (\mu S_t dt)^2 + 2(\mu S_t dt)(\sigma S_t dW_t) + (\sigma S_t dW_t)^2 = \sigma^2 S_t^2 dt$$

(4) Substituting in $(dS_t)^2 = \sigma^2 S_t^2 dt$ and $dS_t = \mu S_t dt + \sigma S_t dW_t$, the equation becomes

$$dV = \Big(\frac{\partial V}{\partial t} + \mu\, S_t\, \frac{\partial V}{\partial S} + \tfrac{1}{2}\,\sigma^2\, S_t^2\, \frac{\partial^2 V}{\partial S^2}\Big)\, dt + \sigma\, S_t\, \frac{\partial V}{\partial S}\, dW_t$$

(5) We proceed by creating a portfolio consisting of a call option, V, with Δ shares of stock. The total value of the portfolio can be represented by: $\Pi = V + \Delta S_t$

(6) To understand how this portfolio moves over time,

$$d\Pi = d\big(V + \Delta S_t\big) = \Big(\tfrac{\partial V}{\partial t} + \mu\, S_t\, \tfrac{\partial V}{\partial S} + \tfrac{1}{2}\,\sigma^2\, S_t^2\, \tfrac{\partial^2 V}{\partial S^2} + \Delta\, \mu\, S_t\Big)\, dt + \sigma\, S_t\, \Big(\tfrac{\partial V}{\partial S} + \Delta\Big)\, dW_t$$

(7) The dW_t term is stochastic, and as a result, is the source of randomness and risk. To make our portfolio risk-free, let $\Delta = -\frac{\partial V}{\partial S}$ be used to eliminate the random term.

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\,\sigma^2\, S_t^2\, \frac{\partial^2 V}{\partial S^2}\right) dt.$$

(8) Since the random component has been eliminated, the risk-free portfolio must grow at a risk-free rate r, as the assumption of no arbitrage implies that the portfolio has the risk-free asset. Thus, the change in the portfolio is

$$d\Pi = r\Pi dt = r(V + \Delta S_t)dt = r(V - S_t\frac{\partial V}{\partial S})dt$$

(9) Substituting in $d\Pi = r(V - S_t \frac{dV}{dS})$ and simplifying, we get the Black-Scholes equation

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} - rV = 0$$

Solving the Black Scholes PDE

We begin with the Black–Scholes partial differential equation and terminal condition:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + r\, S_t \frac{\partial V}{\partial S_t} - r\, V = 0, \qquad V(S_T, T) = \max\bigl(S_T - K, 0\bigr).$$

(1) Backwards time transforms

$$\tau = T - t, \quad \frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau}.$$

The PDE becomes

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + r\, S_t \frac{\partial V}{\partial S_t} - r\, V.$$

(2) The Log-price change of variable.

$$x = \ln\bigl(\tfrac{S_t}{K}\bigr), \quad S_t = K\, e^x, \quad v(x, \tau) = \frac{V(S_t, t)}{K}.$$

By the chain rule:

$$\frac{\partial V}{\partial S_t} = \frac{1}{S_t}\frac{\partial v}{\partial x}, \quad \frac{\partial^2 V}{\partial S_t^2} = \frac{1}{S_t^2}\Bigl(v_{xx} - v_x\Bigr).$$

Substitution yields

$$v_\tau = \frac{1}{2}\sigma^2 (v_{xx} - v_x) + r\, v_x - r\, v.$$

(3) Remove drift and zeroth-order terms so we can get to the heat equation, which is a PDE with a known solution.

$$v(x, \tau) = e^{\,ax + b\tau}\, u(x, \tau).$$

Choosing

$$a = -\frac{r - \frac{1}{2}\sigma^2}{\sigma^2}, \quad b = -\frac{\bigl(r + \frac{1}{2}\sigma^2\bigr)^2}{2\,\sigma^2}$$

cancels the u_x and u terms, leaving the heat equation.

(4) The Heat equation.

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = e^{-ax}\, \max\bigl(e^x - 1, 0\bigr).$$

(5) Heat-kernel solution is:

$$u(x,\tau) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\,\sigma^2\,\tau}} \exp\!\Big(-\frac{(x-y)^2}{2\,\sigma^2\,\tau}\Big)\, u(y,0)\,dy.$$

(6) Revert transforms.
First set $v = e^{ax+b\tau}u$, then $V = K\,v$, and finally $x = \ln(S_t/K)$, $\tau = T-t$. Evaluate the Gaussian integrals in terms of the normal CDF $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy$.

(7) Evaluate the convolution integrals.
From Step 6 we have

$$V(S_t,t) = K\,e^{ax+b\tau} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\,\sigma^2\,\tau}} \exp\!\Big(-\frac{(x-y)^2}{2\sigma^2\tau}\Big)\, e^{-ay}\ \max(e^y - 1, 0)\,dy,$$

where $x = \ln(S_t/K)$ and $\tau = T-t$. Since $\max(e^y - 1, 0) = 0$ for $y \le 0$ and $e^y - 1$ for $y > 0$, split the integral:

$$V = \int_0^{\infty} e^y\, w(y)\,dy \ - \ \int_0^{\infty} w(y)\,dy =: I_1 - I_2,$$

with

$$w(y) = \frac{K}{\sqrt{2\pi\,\sigma^2\,\tau}} \exp\!\Big(a(x-y) + b\tau - \tfrac{(x-y)^2}{2\sigma^2\tau}\Big).$$

(8) For I_1, set

$$z = \frac{y - \big(x + (r + \frac{1}{2}\sigma^2)\tau\big)}{\sigma\sqrt{\tau}},$$

which shows

$$I_1 = S_t \int_{-\infty}^{d_1} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz = S_t\,\Phi(d_1).$$

(9) For I_2, set

$$z = \frac{y - \big(x + (r - \frac{1}{2}\sigma^2)\tau\big)}{\sigma\sqrt{\tau}},$$

giving

$$I_2 = K\,e^{-r\tau} \int_{-\infty}^{d_2} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz = K\,e^{-r\tau}\,\Phi(d_2).$$

(10) Final Black Scholes formula.

$$V(S_t,t) = I_1 - I_2 = S_t\,\Phi(d_1) - K\,e^{-r\,(T-t)}\,\Phi(d_2),$$

where

$$d_{1,2} = \frac{\ln(S_t/K) + \big(r \pm \frac{1}{2}\sigma^2\big)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

A more in depth solution can be found by Dunbar, S (n.d) *Deriving the Black-Scholes solution, University of Nebraska-Lincoln.*

5. Applications

This last section will be used to highlight the applications of the Black-Scholes model. More specifically, we will explore how Black-Scholes and Monte Carlo simulations connect through price option estimates.

Definition 4.2 A *Monte Carlo Simulation* is an algorithm that uses repeated random sampling to obtain the probability of a range of results occurring.

Conceptually, a Monte Carlo Simulation is just a possible path that can occur. For example, let us say that the current stock of Apple is 200 dollars. A possible path the stock could have is that it increases 5 dollars tomorrow and falls 3 dollars the day after. Another possible path the stock could have is that it decreases 5 dollars tomorrow and then increases 10 dollars the day after. By randomly generating thousands of such paths, we can eventually build a probability distribution of what the stock price could be tomorrow. Using these distributions and the number of simulations made, we can find an estimate of the option price, given by te formula:

$$\hat{C} \;=\; e^{-rT}\,\frac{1}{M}\sum_{i=1}^{M}\max\bigl(S_T^{(i)}-K,\;0\bigr),$$

A quick proof of the formula is shown.

(1) Under a risk-neutral measure $\mathbb{Q}$, with no arbitrage, the price of a European call is:

$$C = e^{-rt}\mathbb{E}_{\mathbb{Q}}[(S_T - K, 0]$$

(2) We can approximate $\mathbb{E}_{\mathbb{Q}}[.]$ by simulating random independent samples $S_T^{(1)}, ..., S_T^{(M)}$

(3) Using the law of large numbers,

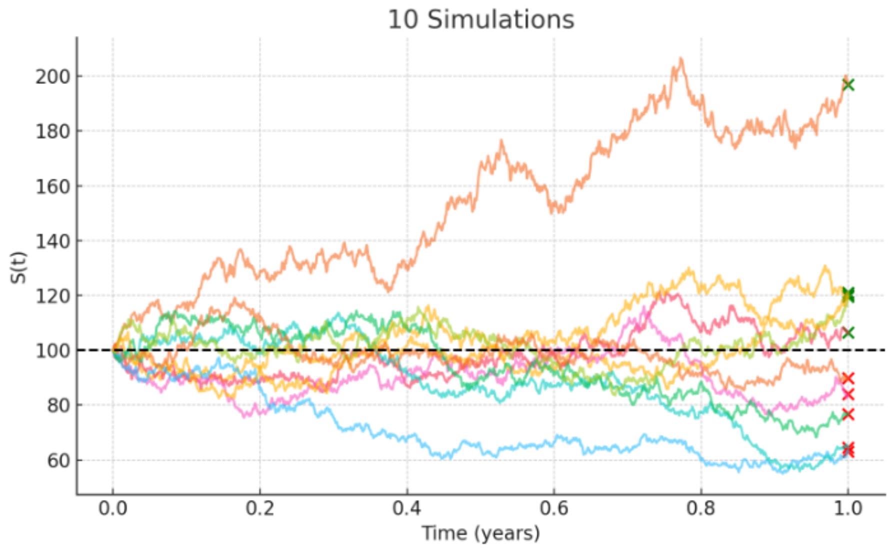
$$\frac{1}{M}\sum_{i=1}^{M}\max\bigl(S_T^{(i)}-K,0\bigr) \;\xrightarrow{M\to\infty}\; \mathbb{E}_{\mathbb{Q}}\bigl[\max(S_T-K,0)\bigr].$$

(4) Rearranging the terms, we get that

$$\hat{C} = e^{-rT}\,\frac{1}{M}\sum_{i=1}^{M}\max\bigl(S_T^{(i)}-K,0\bigr)$$

Now, let us explore our Monte Carlo simulations and its formulas with the following example: We have the parameters that the initial stock price $S_0 = 100$, the strike price $K = 100$, the volatility is $\sigma = 0.3$, time to maturity is $T = 1$ year, and our risk-free rate $r = 0.05$. What should be our option price?

To tackle this question, one method is to utilize Monte Carlo Simulations with varying number of paths. To begin, we can use 10 paths.



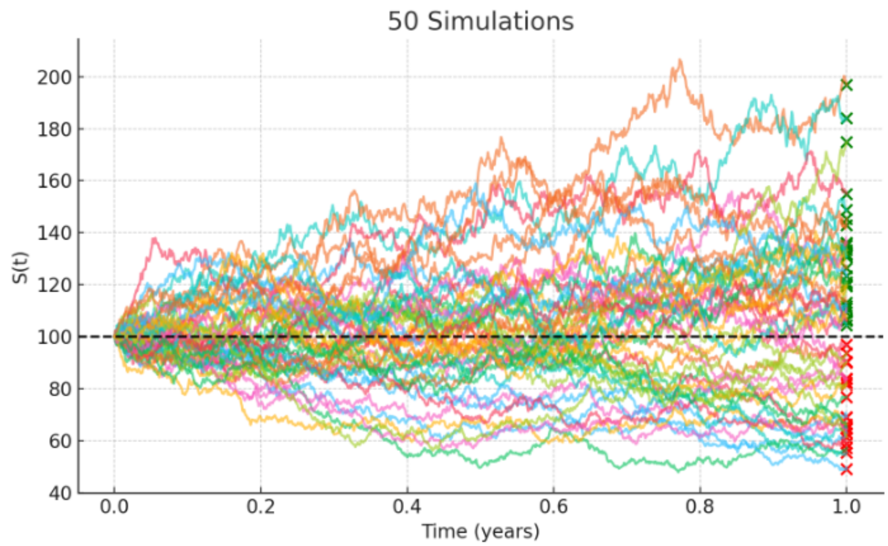
These are all possible paths that the stock price could take, where the small green x marks show a ending price higher than the strike price, and the small red x marks show an ending price less than the strike price.

Using our formula for an estimate of a Monte Carlo Simulation, we see that:

$$C_{\mathrm{MC}} = e^{-rT}\frac{1}{M}\sum_{i=1}^{M}\max\big(S_T^{(i)} - K, 0\big) \implies C_{\mathrm{MC}} = e^{-0.05\cdot 1}\frac{1}{10}\sum_{i=1}^{10}\max\big(S_T^{(i)} - 100, 0\big).$$

Simplifying, we get that the option price using 10 paths should be approximately 5.91 dollars.

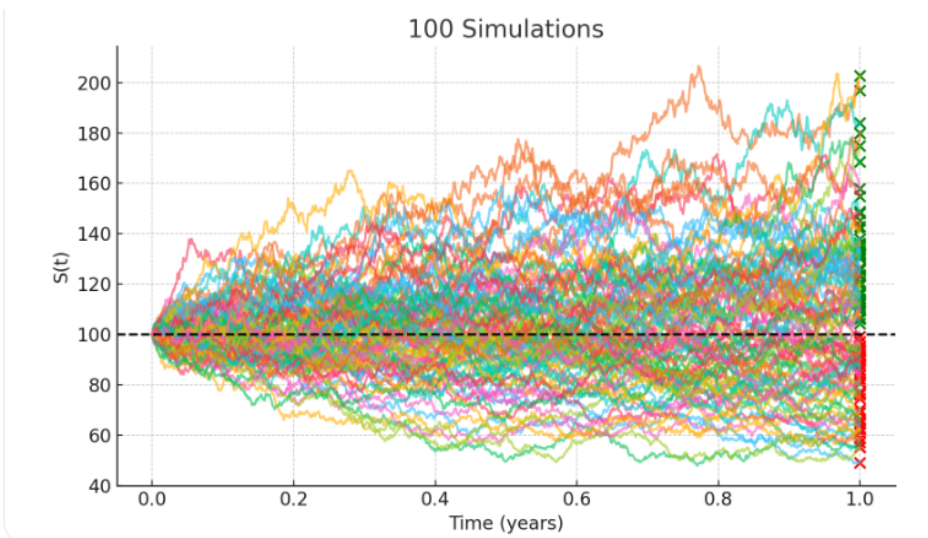
Now, let us say that we want to hae a better understanding of the possible outcomes that the stock price could take. Thus, we use 50 paths in our simulation.



With 50 paths, we can now see many more possibilities. Using our Monte Carlo price estimate formula, we can once again calculate the option price:

$$C_{\text{MC}} \;=\; e^{-rT}\,\frac{1}{50}\sum_{i=1}^{50}\max\bigl(S_T^{(i)}-100,\,0\bigr)\;\approx\;16.43.$$

Finally, we can run the simulation one last time with 100 paths.



Our formula shows the option price:

$$C_{\text{MC}} \;=\; e^{-rT}\,\frac{1}{100}\sum_{i=1}^{100}\max\bigl(S_T^{(i)}-100,\,0\bigr)\;\approx\;12.83.$$

If each scenario gives us a different option price, which is the true value? Well, the true option price is actually none of them. Rather, the true option price is given as we approach infinitely many paths. However, it is expensive and time taking to create millions of paths, making it difficult to find the true option price. This is where the Black-Scholes pricing equation becomes useful. Instead of simulating millions and billions of paths, we can simply plug in the necessary values into the pricing equation. Mathematically, we can represent the connection between the Black-Scholes pricing equation and the Monte Carlo estimate as:

$$\lim_{M\to\infty} C_{\mathrm{MC}}(M) \;=\; C_{\mathrm{BS}} \;=\; S_0\,\Phi(d_1) \;-\; Ke^{-rT}\,\Phi(d_2).$$

Using this formula, and the parameters outlined on page 14, we can find the true value of the option:

$$\begin{aligned}
C_{\mathrm{BS}} &= S_0\,\Phi(d_1) \;-\; K\,e^{-rT}\,\Phi(d_2),\\
d_1 &= \frac{\ln\bigl(S_0/K\bigr) + \bigl(r+\tfrac{1}{2}\sigma^2\bigr)T}{\sigma\sqrt{T}} = \frac{0+(0.05+0.5\cdot 0.09)\cdot 1}{0.3} = \frac{0.095}{0.3} \approx 0.3167,\\
d_2 &= d_1 - \sigma\sqrt{T} \approx 0.3167 - 0.3 = 0.0167,\\
\Phi(d_1) &\approx 0.6246, \quad \Phi(d_2) \approx 0.5067, \quad e^{-rT} = e^{-0.05} \approx 0.9512,\\
C_{\mathrm{BS}} &\approx 100\cdot 0.6246 \;-\; 100\cdot 0.9512\cdot 0.5067 \;\approx\; 62.46 - 48.20 \;=\; 14.26.
\end{aligned}$$

With that, we have now found the connection between the Black-Scholes model and Monte Carlo simulations.

Acknowledgments. First, I would like to thank my mentor, Daniel Naylor, for all the support in the process of writing this paper. Specifically, I appreciate all the resources, guidance, and encouragement provided. Second, I would like to thank director Simon Rubinstein Salzedo for the opportunity to learn at Euler Circle. The class allowed me to explore a topic I'm interested in, and was truly a unique experience.

References

[1] Geoffrey Grimmett and David Stirzaker, *Probability and Random Processes*, 3rd ed., Oxford University Press, 2001.

[2] Bernt Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, 5th ed., Springer-Verlag, Heidelberg/New York, 2003.

[3] Martin Haugh, *The Black–Scholes Model*, IEOR E4706: Foundations of Financial Engineering Lecture Notes, Columbia University, 2016. Available at `https://www.columbia.edu/~mh2078/FoundationsFE/BlackScholes.pdf`

[4] Younggeun Yoo, "Stochastic Calculus and Black–Scholes Model," REU 2017 paper, University of Chicago, July 16, 2017. Available at `https://math.uchicago.edu/~may/REU2017/REUPapers/Yoo.pdf`

[5] Fischer Black and Myron Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, vol. 81, no. 3, pp. 637–654, 1973. Available at `https://www.cs.princeton.edu/courses/archive/fall09/cos323/papers/black_scholes73.pdf`

[6] Scott Guernsey, "An Introduction to the Black–Scholes PDE Model," REU proposal, University of New Mexico, 2013. Available at `https://math.unm.edu/~nitsche/mctp/reus/proposals/2013prop_Guernsey.pdf`

[7] Sheldon M. Ross, *An Elementary Introduction to Mathematical Finance: Options and Other Topics*, Cambridge University Press, 2002. Available at `https://catdir.loc.gov/catdir/samples/cam033/2002073603.pdf`

[8] Tomas Björk, *Arbitrage Theory in Continuous Time*, 2nd ed., Oxford University Press, 2009.