The Lindström-Gessel-Viennot Lemma

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Euler Circle

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Permutations And Signs

Permutations

Definition 1.1

A *permutation* is a bijective function σ where $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

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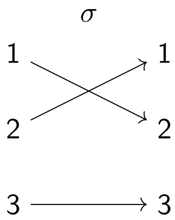

Figure 1: A permutation on $\{1,2,3\}$.

Swaps

Definition 1.2

A permutation is a swap if it swaps two elements and keeps the rest fixed. It is denoted π_{ij} where i and j are the elements we are swapping.

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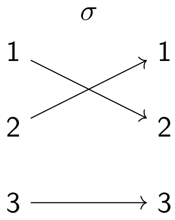

Figure 2: The swap π_{12} on $\{1,2,3\}$.

Composition Of Swaps

Proposition 1.3

Every permutation can be expressed as a composition of swaps.

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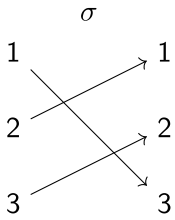

Figure 3: $\sigma = \pi_{12} \circ \pi_{13}$.

Sign of A Permutation

Definition 1.4

Given a permutation σ, we define sign(σ) as

$$\text{sign}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is composed by an even \# of swaps} \\ -1 & \text{if } \sigma \text{ is composed by an odd \# of swaps} \end{cases}$$

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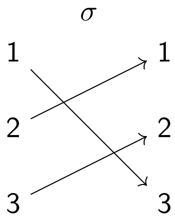

Figure 4: Since $\sigma = \pi_{12} \circ \pi_{13}$ it is even.

Graphs

Ant Problem

Question 2.1

If an ant can only move to the right along the grid, how many ways are there for the ants in Figure 5 to reach different pieces of food without their paths crossing each other?

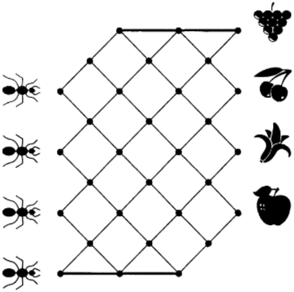

Figure 5: Ants and Morsels.

Basic Assumptions

The LGV-Lemma works with graphs that are finite, weighted, acyclic, and directed. We also take two groups of n (not necessarily disjoint) vertices. They are usually denoted $\mathcal{A} = \{A_1, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, \ldots, B_n\}$. In practice, $\mathcal{A}$ is our set of origins, and $\mathcal{B}$ is the set of destinations.

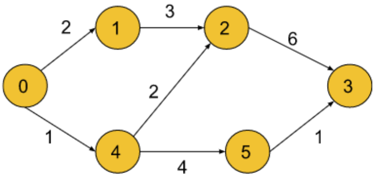

Figure 6: Weighted acyclic directed graph.

Path Weights

Definition 2.2

Suppose we have a path P between vertices $A \in \mathcal{A}$ and $B \in \mathcal{B}$, denoted $P : A \to B$, then we define its weight as

$$w(P) = \prod_{e \in P} w(e).$$

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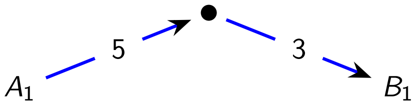

Figure 7: $w(P : A_1 \to B_1) = 5 \cdot 3 = 15.$

Path Matrices

Definition 2.3

We can define the path matrix M with respect to $\mathcal{A}$ and $\mathcal{B}$ as having

$$m_{ij} = \sum_{P:A_i \to B_j} w(P).$$

If there do not exist any paths between A_i and B_j then $m_{ij} = 0$.

Path Matrices

Example

The path matrix for the graph depicted in Figure 8 is

$$\begin{bmatrix} 33 & 0 \\ 0 & 7 \end{bmatrix}.$$

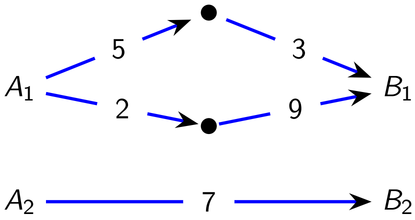

Figure 8: A graph with $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{B} = \{B_1, B_2\}$.

Path Systems

Definition 2.4

A *path system* from $\mathcal{A}$ to $\mathcal{B}$ consists of a permutation σ together with n paths $P_i : A_i \to B_{\sigma(i)}$, for $i = 1, \ldots, n$.

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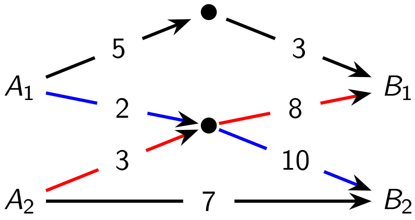

Figure 9: A graph with $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{B} = \{B_1, B_2\}$. P_1 is the blue path and P_2 is the red path. The permutation associated with the path system is π_{12}.

Weight Of A Path System

Definition 2.5

We define the weight of a path system as the product of its path weights. That is,

$$w(\mathcal{P}) = \prod_{i=1}^{n} w(P_i)$$

where the P_i are the paths associated with the path system.

Weight Of A Path System

Example

The weight of the path system depicted in Figure 10 is

$$w(P_1) \cdot w(P_2) = 20 \cdot 24 = 480.$$

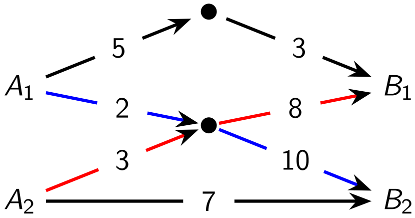

Figure 10: A graph with $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{B} = \{B_1, B_2\}$.$P_1$ is the blue path and P_2 is the red path. The permutation associated with the path system is π_{12}.

Final Definitions

Definition 2.6

We define sign($\mathcal{P}$) as being equal to sign(σ) where σ is the permutation associated with $\mathcal{P}$.

Definition 2.7

We say a path system $\mathcal{P} = (P_1, \ldots, P_n)$ is vertex disjoint if the paths of $\mathcal{P}$ are pairwise disjoint.

The Lindström-Gessel-Viennot Lemma

Lemma 1 (Lindström-Gessel-Viennot)

Let $G = (V, E)$ be a finite weighted acyclic directed graph, $\mathcal{A} = \{A_1, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, \ldots, B_n\}$ two sets of n vertices, and M the path matrix from $\mathcal{A}$ to $\mathcal{B}$. Then

$$\det(M) = \sum_{\substack{\mathcal{P} \text{ vertex-disjoint} \\ \text{path system}}} \operatorname{sign}(\mathcal{P})\, w(\mathcal{P}).$$

Interesting Applications

Application

How do we use the LGV-Lemma in practice? This lemma is incredibly adaptable since you can either solve a determinant to get an answer to a combinatorics problem, or you can analyze a combinatorics problem to solve for a determinant.

Ant Problem

Question 3.1

If an ant can only move to the right along the grid, how many ways are there for the ants in Figure 11 to reach different pieces of food without their paths crossing each other?

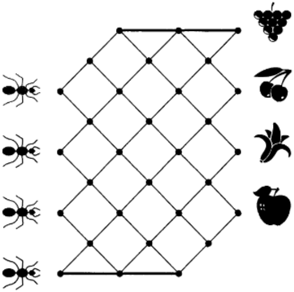

Figure 11: Ants and Morsels.

Is The LGV-Lemma Applicable?

Yes! Because the ants can only go to the right, the graph has direction. It is also easy to see that the graph is acyclic. We also have natural choice for our two sets of vertices. The 4 origin points (the ants) and the 4 destinations (the food).

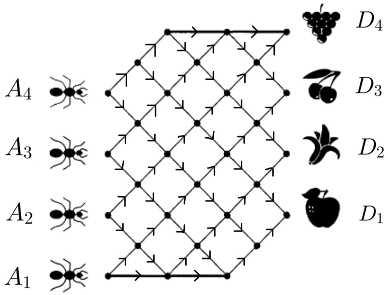

Figure 12: Ants and Morsels.

Computation Of The Path Matrix

We set the weight of each edge to be 1. This makes each path have weight 1 making the m_{ij} entry of the path matrix to be the number of paths between A_i and D_j. How do we count this? Using recursion!

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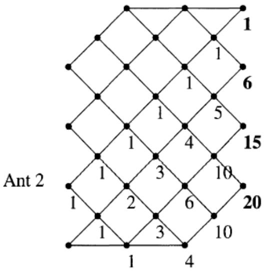

Figure 13: Recursively counting paths for Ant 2.

Computation Of The Path Matrix

We obtain the path matrix $M = \begin{bmatrix} 14 & 6 & 1 & 0 \\ 20 & 15 & 6 & 1 \\ 15 & 20 & 15 & 6 \\ 6 & 20 & 15 & 14 \end{bmatrix}$.

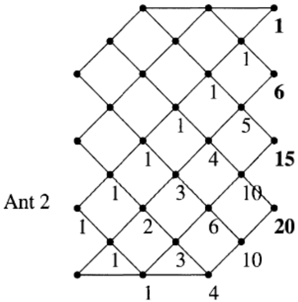

Figure 14: Recursively counting paths for Ant 2.

Nonpermutablility

Question 3.2

Why is the fact that

$$\det(M) = \sum_{\substack{\mathcal{P} \text{ vertex-disjoint} \\ \text{path system}}} \mathit{sign}(\mathcal{P})\, w(\mathcal{P})$$

useful?

Nonpermutablility

Observation 3.3

The only vertex-disjoint path systems are ones that send A_i to D_i!

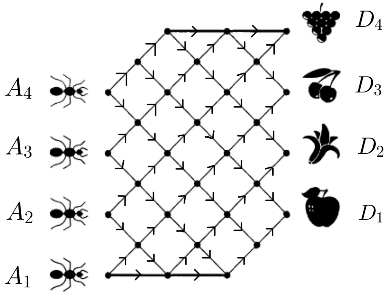

Figure 15: Ants and Morsels.

Nonpermutability

Definition 3.4

graph $G = (V, E)$ that satisfies the conditions of the LGV-Lemma is called *nonpermutable* if all vertex-disjoint path systems are associated with the identity permutation.

Nonpermutability

Corollary 3.5

If a graph $G = (V, E)$ satisfies the conditions for the LGV-Lemma, is nonpermutable, and all its edges have weight 1, then

$$\det(M) = \# \text{ of vertex-disjoint path systems} .$$

Proof.

Since G is nonpermutable, we know that all its vertex-disjoint path systems have sign equal to 1. Additionally, the weight of all path systems is 1. Thus, we get

$$\det(M) = \sum_{\substack{\mathcal{P} \text{ vertex-disjoint} \\ \text{path system}}} \text{sign}(\mathcal{P})\, w(\mathcal{P}) = \sum_{\substack{\mathcal{P} \text{ vertex-disjoint} \\ \text{path system}}} 1$$

The Answer!

We can calculate the determinant of our path matrix M to be $\det(M) = 889$ using a Laplace Expansion. It follows directly from Corollary 3.5 that the number of non-intersecting path systems is in fact 889.

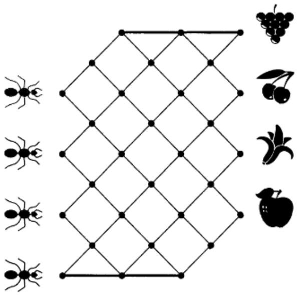

Figure 16: Ants and Morsels.

Another Application

Theorem 3.6 (Binet-Cauchy)

If P is a $r \times s$ matrix and Q an $s \times r$ matrix, $r \leq s$, then

$$\det(PQ) = \sum_{\mathcal{Z}} (\det P_{\mathcal{Z}})(\det Q_{\mathcal{Z}}),$$

where $P_{\mathcal{Z}}$ is the $r \times r$ submatrix of P with column-set $\mathcal{Z}$, and $Q_{\mathcal{Z}}$ the $r \times r$ submatrix with the corresponding rows $\mathcal{Z}$.