Pell's Equation:
A Gateway to Number Theory and Continued Fractions

Ryan Kim

Euler Circle

July 11, 2025

Contents

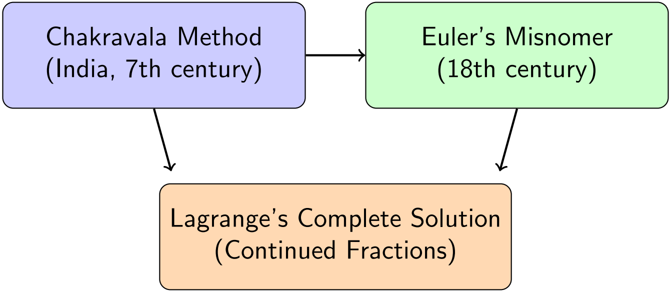
Ancient Roots

- 7th-century India: Brahmagupta and Bhaskara's Chakravala method
- Efficient cyclic algorithm for solving $x^2 - dy^2 = 1$
- Remarkably accurate without modern algebra

A Misnamed Equation

- Euler credited John Pell incorrectly
- Lagrange (18th century): first complete solution via continued fractions
- Despite the name, Pell had little to do with it!

Timeline Overview



What is Pell's Equation?

$$x^2 - dy^2 = 1$$

- d is a fixed positive integer that is **not** a perfect square
- Find all integer solutions (x, y)
- Infinitely many solutions exist!

Why is it Hard?

- Solutions are not obvious, unlike quadratic equations
- Small values of d yield huge fundamental solutions
- Deep connection to irrationality of $\sqrt{d}$

Example: $d = 2$

- $x^2 - 2y^2 = 1$
- Solutions: $(3, 2), (17, 12), (99, 70), \ldots$
- Generated by powers of $3 + 2\sqrt{2}$

Statement of Theorem

Lagrange's Solution Theorem:
For any non-square d, the equation

$$x^2 - dy^2 = 1$$

has a positive integer solution x, y arising from a convergent in the continued fraction of $\sqrt{d}$.

Continued Fraction Setup

Let $\sqrt{d} = [a_0; \overline{a_1, \ldots, a_k}]$.
Let $\frac{p_n}{q_n}$ be the n-th convergent.
Define $\delta_n = p_n^2 - dq_n^2$.

We aim to show:

$$\delta_n = \pm 1$$

for some n.

Periodicity and Symmetry

- Lagrange's theorem: $\sqrt{d}$ has periodic continued fraction
- Even period: $\frac{p_{k-1}}{q_{k-1}}$ gives a solution
- Odd period: $\frac{p_{2k-1}}{q_{2k-1}}$ does

This symmetry ensures $\delta_n = \pm 1$ at one of these positions.

Why This Works

$$\left|\sqrt{d} - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$$

Multiply both sides by $q_n\sqrt{d}$ and rearrange:

$$p_n^2 - dq_n^2 \approx 0$$

Since this quantity is always an integer, it must eventually equal ± 1.

Conclusion of Proof

$$p_n^2 - dq_n^2 = \pm 1 \Rightarrow (p_n, q_n)$$

is a solution.

This gives the minimal solution (x, y).

- All others from powers of $x + y\sqrt{d}$:

$$(x + y\sqrt{d})^n = x_n + y_n\sqrt{d}$$

Example: $\sqrt{13}$

- $\sqrt{13} = [3; \overline{1, 1, 1, 1, 6}]$
- 4th convergent: $\frac{649}{180}$
- $649^2 - 13 \cdot 180^2 = 1$

So $(649, 180)$ is the minimal solution.

Algebraic Number Theory Connection

- Pell's equation = finding units of norm 1 in $\mathbb{Z}[\sqrt{d}]$
- Dirichlet's Unit Theorem: unit group is infinite cyclic
- Fundamental unit generates all others

Geometric Interpretation

- $x^2 - dy^2 = 1$ is a hyperbola
- Integer points on this hyperbola are rare and structured
- Solutions lie on discrete lattice intersecting the curve

Pell Modulo p

- Pell's equation mod p gives residue constraints
- Not always solvable modulo a given prime
- Helps study solution density across d

Surprise: Enormous Fundamental Solutions

- For some d, fundamental solution is astronomically large
- Example: $d = 61 \Rightarrow x = 1766319049$
- No shortcut — CF still works

Wrap-Up

- From ancient math to modern algebraic number theory
- Continued fractions are the gateway
- Pell's equation connects everything from irrationality to units in rings

Thank you!

Thank you!