Continuous Binomial Coefficients

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Euler Circle

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Extending the Factorials

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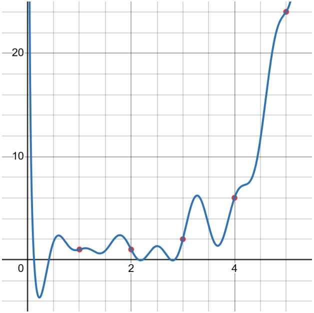
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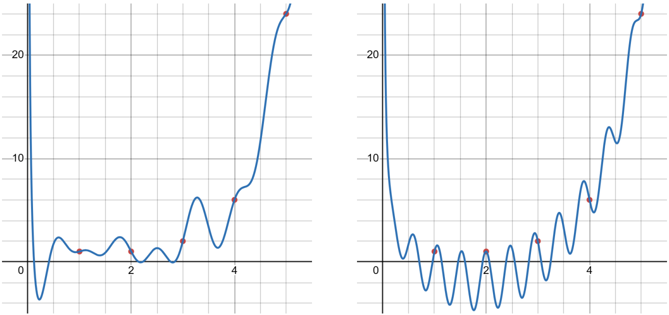
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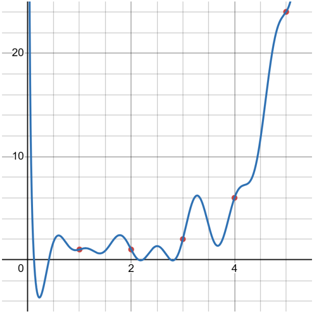
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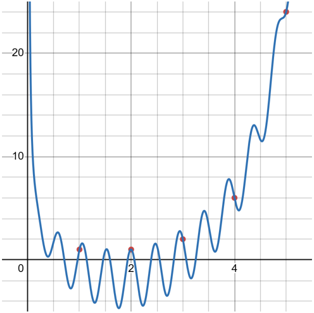
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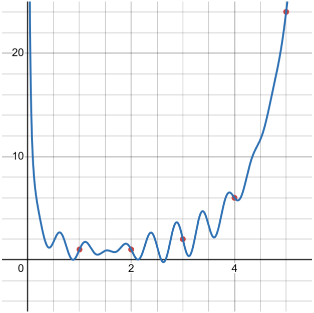
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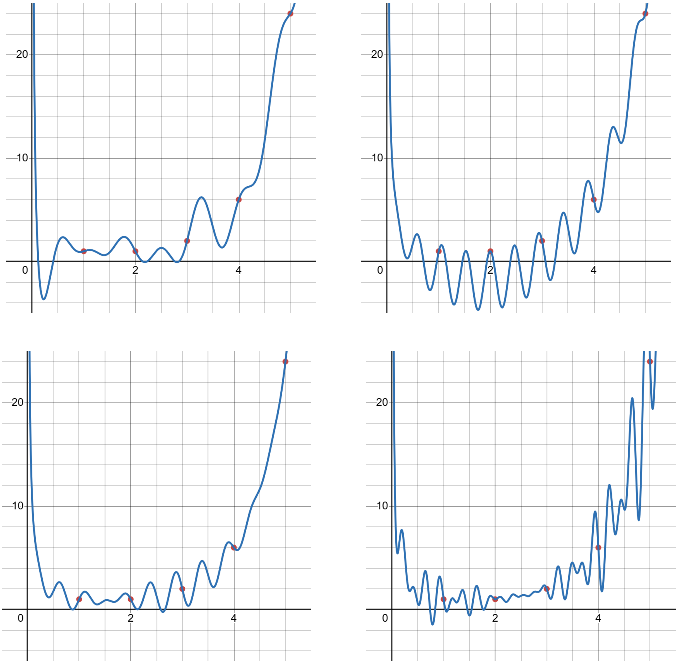
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1. $\Gamma(1) = 1.$
2. $\Gamma(z+1) = z\Gamma(z).$
3. $\frac{d^2}{d^2 z} \ln(\Gamma(z)) > 0$

The Bohr-Mollerup Theorem

Theorem (Bohr–Mollerup Theorem)

There exists a unique function $\Gamma(z)$ *which satisfies the following properties:*

1. $\Gamma(z+1) = z\Gamma(z)$
2. $\Gamma(1) = 1$
3. $\frac{d^2}{d^2 z} \ln(\Gamma(z)) > 0$

The Bohr-Mollerup Theorem

Proof.

Let $S(x_1, x_2)$ be the slope of the secant line of $\ln \Gamma(x)$ between x_1 and x_2.

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$$S(x_1, x_2) = \frac{\ln \Gamma(x_2) - \ln \Gamma(x_1)}{x_2 - x_1} = \frac{\ln \left(\Gamma(x_2)/\Gamma(x_1)\right)}{x_2 - x_1}. \tag{0.1}$$

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$S(x_1, x_2)$ is monotonically increasing by the Log Convexity of $\Gamma(x)$, so for $x \in (0, 1]$ and $n \in \mathbb{Z}^+$ we have

$$S(n-1, n) \leq S(n, n+x) \leq S(n, n+1). \tag{0.2}$$

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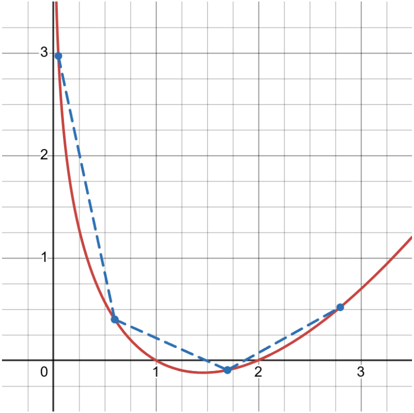

Figure: Plot of $\ln \Gamma(x)$

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Substituting (0.1) into (0.2) and exponentiating, we get

$$(n-1)^x (n-1)! \leq \Gamma(n+x) \leq (n)^x (n-1)!. \tag{0.3}$$

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To complete the proof, rearrange (0.3) for an inequality of $\Gamma(x)$ and take the limit as $n \to \infty$ to sandwich $\Gamma(x)$. The computation is omitted. ∎

The Gamma Function

Definition

$$\Gamma(z+1) \equiv \lim_{n\to\infty} n^z \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right)^{-1}.$$

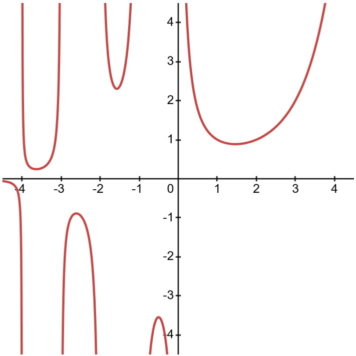

Figure: Plot of $\Gamma(z)$

The Gamma Function

Definition (Euler's Definition)

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$$\Gamma(z) \equiv \int_0^{\infty} t^{z-1}e^{-t}dt$$

Definition (Weierstrass' Definition)

$$\Gamma(z) \equiv \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\frac{e^{z/n}}{1+\frac{z}{n}}.$$

Where γ is the *Euler-Mascheroni constant*.

Continuous Binomial Coefficients

Discrete Binomial Coefficients

Recall that Binomial Coefficients are traditionally defined as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{0.4}$$

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We can extend (0.4) using the Gamma function:

Definition (Continuous Binomial Coefficients)

$$\binom{y}{x} = \frac{\Gamma(y+1)}{\Gamma(x+1)\Gamma(y-x+1)}.$$

A Fun Plot

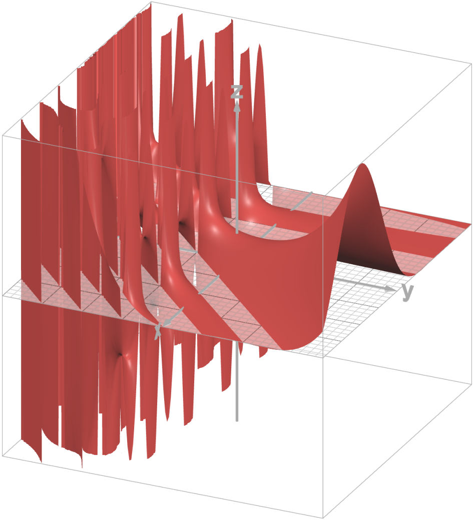

Figure: Plot of $\binom{y}{x}$ on $[-4, 4] \times [-4, 4]$

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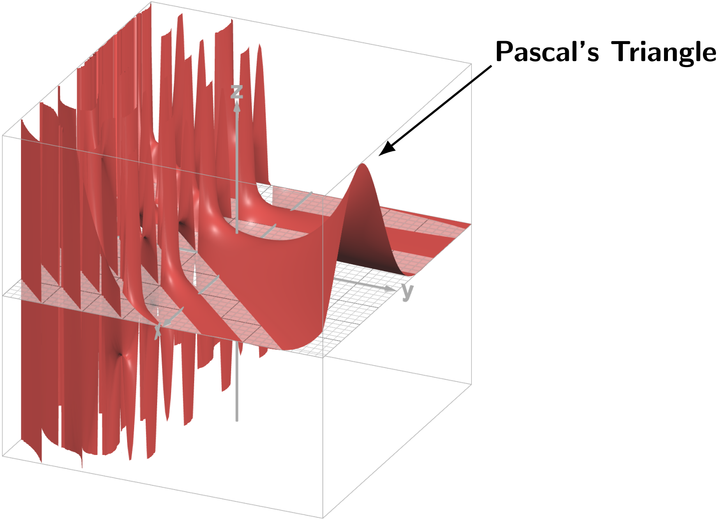

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Theorem (Discrete Binomial Theorem)

For non-negative integers n and k,

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The continuous Binomial Coefficients satisfy a similar theorem.

Theorem (Generalized Binomial Theorem)

For $y \in \mathbb{R}$ and $x \in (-1, 1)$,

$$(1+x)^y = \sum_{n=0}^{\infty} \binom{y}{n} x^n.$$

Continuous Binomial Coefficients

Proof.

Start by writing the Maclaurin series of $(1+x)^y$. For $|x| < 1$,

$$(1+x)^y = \sum_{n=0}^{\infty} \frac{y(y-1)\cdots(y-n+1)}{n!} x^n. \tag{0.5}$$

From $\Gamma(z+1) = z\Gamma(z)$, we have $y(y-1)\cdots(y-n+1) = \frac{\Gamma(y+1)}{\Gamma(1+y-n)}$. Therefore (0.5) can be written as

$$(1+x)^y = \sum_{n=0}^{\infty} \binom{y}{n} x^n.$$

∎

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$$(1+x)^y = x(1+x^{-1})^y = \sum_{n=0}^{\infty} \binom{y}{n} x^{y-n}.$$

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Remark

The Generalized Binomial Theorem also holds for $x = 1$ if $y > -1$ and for $x = -1$ if $y > 0$.

Binomial Identities

The following fundamental properties of discrete binomial coefficients also hold in the continuous case:

Symmetry

$$\binom{y}{x} = \binom{y}{y-x}. \tag{0.6}$$

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Recursion(Pascal's Rule)

$$\binom{y}{x} = \binom{y-1}{x-1} + \binom{y-1}{x}. \tag{0.8}$$

More Binomial Identities

Generalized Hockey-Stick Identity

$$\sum_{i=0}^{n} \binom{i}{x} = \binom{n+1}{x+1} - \binom{0}{x+1} \tag{0.9}$$

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Generalized Row Summation Identity

$$\sum_{k=0}^{\infty} \binom{y}{k} = 2^y, \qquad y > -1. \tag{0.10}$$

Product Representation

Theorem (Product Representation)

$$\binom{y}{\frac{y}{2}+x} = \binom{y}{\frac{y}{2}} \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{(k+\frac{y}{2})^2}\right).$$

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Proof.

By the limit definition of $\Gamma(x)$, we have

$$\Gamma\left(1+\frac{y}{2}+x\right) = \lim_{n\to\infty} n^{\frac{y}{2}+x} \prod_{k=1}^{n} \left(1+\frac{\frac{y}{2}+x}{k}\right)^{-1}.$$

And

$$\Gamma\left(1+\frac{y}{2}-x\right) = \lim_{n\to\infty} n^{\frac{y}{2}-x} \prod_{k=1}^{n} \left(1+\frac{\frac{y}{2}-x}{k}\right)^{-1}.$$

Product Representation

Proof.

Multiplying the expressions and simplifying, we get

$$\Gamma\left(1+\frac{y}{2}-x\right)\Gamma\left(1+\frac{y}{2}+x\right)=\frac{\Gamma\left(1+\frac{y}{2}\right)^2}{\prod_{k=1}^{\infty}\left(1-\frac{x^2}{(k+\frac{y}{2}^2)}\right)}$$

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After rearranging, we arrive at

$$\binom{y}{\frac{y}{2}+x}=\binom{y}{\frac{y}{2}}\prod_{k=1}^{\infty}\left(1-\frac{x^2}{(k+\frac{y}{2})^2}\right).$$

■

Integral Representation

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Proof.

Applying integration by parts twice to the RHS and rearranging, we get

$$\int_0^{\frac{\pi}{2}} \cos^y(t)\cos(2xt)dt = \frac{y+2}{y+1}\left(1-\frac{x^2}{\left(1+\frac{y}{2}\right)^2}\right)\int_0^{\frac{\pi}{2}} \cos^{y+2}(t)\cos(2xt)dt.$$

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$$\int_0^{\frac{\pi}{2}} \cos^y(t)\cos(2xt)dt = \frac{y+2}{y+1}\left(1-\frac{x^2}{\left(1+\frac{y}{2}\right)^2}\right)\int_0^{\frac{\pi}{2}} \cos^{y+2}(t)\cos(2xt)dt.$$

Applying the recursion n times,

$$\int_0^{\frac{\pi}{2}} \cos^y(t)\cos(2xt)dt$$

$$= \prod_{k=1}^{n}\frac{y+2k}{y+2k-1}\prod_{k=1}^{n}\left(1-\frac{x^2}{\left(1+\frac{y}{2}\right)^2}\right)\int_0^{\frac{\pi}{2}} \cos^{y+2n}(t)\cos(2xt)dt.$$

Integral Representation

Proof.

Computing the limit as $n \to \infty$,

$$\int_0^{\frac{\pi}{2}} \cos^y(t)\cos(2xt)dt$$

$$= \frac{\pi}{2^{y+1}} \lim_{n\to\infty} \frac{\left[(2n)^y \prod_{k=1}^{2n} \left(\frac{y}{k}+1\right)^{-1}\right]}{\left[n^{y/2} \prod_{k=1}^{n} \left(\frac{y}{2k}+1\right)^{-1}\right]^2} \prod_{k=1}^{n} \left(1 - \frac{x^2}{\left(1+\frac{y}{2}\right)^2}\right).$$

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Remembering the limit definition of $\Gamma(z)$, we find

$$\int_0^{\frac{\pi}{2}} \cos^y(t)\cos(2xt)dt = \frac{\pi}{2^{y+1}}\binom{y}{\frac{y}{2}} \prod_{k=1}^{n}\left(1-\frac{x^2}{\left(1+\frac{y}{2}\right)^2}\right).$$

Using the product representation of binomial coefficients and rearranging yields the desired result. ∎

Continuous Binomial Formulas

We now have three different formulas to represent the continuous binomial coefficients.

Gamma Representation

$$\binom{y}{\frac{y}{2}+x} = \frac{\Gamma(y+1)}{\Gamma(\frac{y}{2}+x+1)\Gamma(\frac{y}{2}-x+1)}.$$

Product Representation

$$\binom{y}{\frac{y}{2}+x} = \binom{y}{\frac{y}{2}} \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{(k+\frac{y}{2})^2}\right).$$

Integral Representation

$$\binom{y}{\frac{y}{2}+x} = \frac{2^y}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^y(t)\cos(2xt)dt$$

The Continuous Binomial Theorem

Theorem (Continuous Binomial Theorem)

$$\int_{-\infty}^{\infty} \binom{y}{x} z^x dx = (1+z)^y, \qquad y \in \mathbb{R}^+, \ z \in \mathbb{C}, \ |z| = 1.$$

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Proof.

Let $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Using the product representation we get

$$\int_{-\infty}^{\infty} \binom{y}{\frac{y}{2}+x} e^{i\theta x} dx = \frac{2^y}{\pi} e^{\frac{i\theta y}{2}} \int_{-\infty}^{\infty} e^{i\theta x} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^y(t) \cos(2xt) dt\ dx.$$

Swapping the integrals and computing leads to

$$\int_{-\infty}^{\infty} \binom{y}{\frac{y}{2}+x} e^{i\theta x} dx = \frac{2^y}{\pi} e^{\frac{i\theta y}{2}} \lim_{k \to \infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^y(t)}{t - \frac{\theta}{2}} \sin(2k(t - \theta/2)) dt.$$

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$$\int_{-\infty}^{\infty} \binom{y}{\frac{y}{2}+x} e^{i\theta x} dx = \frac{2^y}{\pi} e^{\frac{i\theta y}{2}} \lim_{k\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^y(t)}{t-\frac{\theta}{2}} \sin(2k(t-\theta/2))dt.$$

Evaluating the above integral yields

$$\int_{-\infty}^{\infty} \binom{y}{\frac{y}{2}+x} e^{i\theta x} dx = \left[2e^{i\frac{\theta}{2}} \cos\left(\frac{\theta}{2}\right)\right]^y \tag{0.11}$$

$$= \left(2\cos^2\left(\frac{\theta}{2}\right) + 2i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)^y \tag{0.12}$$

$$= \left(1+\cos(\theta)+i\sin(\theta)\right)^y = (1+e^{i\theta})^y. \tag{0.13}$$

As desired. ∎

Summary

Definition

$$\Gamma(z+1) \equiv \lim_{n\to\infty} n^z \prod_{k=1}^{n} \left(1+\frac{z}{k}\right)^{-1}.$$

Definition (Continuous Binomial Coefficients)

$$\binom{y}{x} = \frac{\Gamma(y+1)}{\Gamma(x+1)\Gamma(y-x+1)}.$$

Theorem (Generalized Binomial Theorem)

For $y \in \mathbb{R}$ and $x \in (-1,1)$,

$$(1+x)^y = \sum_{n=0}^{\infty} \binom{y}{n} x^n.$$

Summary

Theorem (Product Representation)

$$\binom{y}{\frac{y}{2}+x} = \binom{y}{\frac{y}{2}} \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{(k+\frac{y}{2})^2}\right).$$

Theorem (Integral Representation)

$$\binom{y}{\frac{y}{2}+x} = \frac{2^y}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^y(t)\cos(2xt)dt$$

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