EXTENDING THE BINOMIAL COEFFICIENTS

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Abstract. This paper investigates a continuous extension of the binomial coefficients defined with the Gamma function. We prove several continuous analogues to identities of the classical binomial coefficients. We also investigate the behavior of the derivatives of this extension.

1. Introduction

Euler first developed the Gamma function as an extension to the factorials in a letter to Goldbach in 1729. Euler's solution to the problem of extending the factorials expressed the continuous factorials as an infinite product. He later refined this expression using an integral. The notation $\Gamma(z)$ for the extension of the factorials was first introduced by Legendre in 1809. In this paper, we use this extension to the factorials to define an extension to the binomial coefficients defined in terms of the factorial function. Since the Gamma function obeys the same fundamental recursive property as the factorials, many of the same identities that hold for classical binomial coefficients also work in the continuous case. Furthermore, using the properties of the Gamma function, we show that the continuous binomial coefficients obey several continuous analogues to the discrete identities of the classical binomial coefficients. We also briefly investigate the behavior of the higher order derivatives of the continuous binomial coefficients and show that they obey an infinite family of differential equations.

2. Preliminaries

2.1. **Derivation of the Gamma Function.** The Gamma function was first introduced by Euler as a way to extend the factorial function onto the real plane. It's defined as the unique *logarithmically convex* function with the properties that $\Gamma(1) = 1$ and $\Gamma(z+1) = z\Gamma(z)$. These properties ensure $\Gamma(z+1) = z!$ for z a non-negative integer. Logarithmic convexity simply means $ln\Gamma(z)$ is convex. The requirement of being *analytic*(Being expressible as a convergent Taylor series) is not sufficient to ensure uniqueness since adding periodic analytic functions which are equal to zero at positive integers(e.g. $\sin(\pi z)$) results in another analytic function which interpolates the factorials. The extra requirement of logarithmic convexity can be thought of intuitively as precluding sums with periodic functions.

Theorem 2.1 (Bohr–Mollerup Theorem)**.**
There exists a unique function $\Gamma(z)$ *which satisfies the following properties*

(1) $\Gamma(z+1) = z\Gamma(z)$
(2) $\Gamma(1) = 1$
(3) $\frac{d^2}{d^2z}\ln(\Gamma(z)) > 0$

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Proof. property (1) implies that $\forall n \in \mathbb{N}, \Gamma(z+n) = z(z+1)\cdots(z+n)\Gamma(z)$. Therefore $\Gamma(z)$ on $\mathbb{R}$ is uniquely defined by its value on $(0,1]$. Let $S(z_1, z_2)$ be the slope of the line segment between the points $(z_1, \ln(\Gamma(z_1)))$ and $(z_2, \ln(\Gamma(z_2)))$ where $z_1 < z_2$. By the convexity of $\ln(\Gamma(x))$, $S(z_1, z_2)$ is increasing. Therefore,

$$\forall z \in (0,1], S(n-1, n) \leq S(n, n+z) \leq S(n, n+1). \tag{2.1}$$

Since $S(z_1, z_2) = \frac{\ln\Gamma(z_2) - \ln\Gamma(z_1)}{z_2 - z_1} = \frac{ln(\Gamma(z_2)/\Gamma(z_1))}{z_2 - z_1}$, we can exponentiate (2.1) to get

$$(n-1)^z(n-1)! \leq \Gamma(n+z) \leq (n)^z(n-1)!. \tag{2.2}$$

Note that each inequality is separate and applies for any positive integer n, so we can take $n = n+1$ on the LHS but keep $n = n$ on the RHS. Applying $\Gamma(z+n) = (\prod_{k=0}^{n} z+k)\Gamma(z)$ and rearranging, we get

$$\left(\frac{n}{n+z}\right)\frac{(n)^z(n-1)!}{z(z+1)\cdots(z+n-1)} \leq \Gamma(z) \leq \frac{(n)^z(n-1)!}{z(z+1)\cdots(z+n-1)}. \tag{2.3}$$

Since these inequalities are true for any positive integer n, we can take the limit as $n \to \infty$. Using $\lim_{n\to\infty}\left(\frac{n}{n+z}\right) = 1$, (2.3) becomes

$$\lim_{n\to\infty}\frac{(n)^z(n-1)!}{z(z+1)\cdots(z+n-1)} \leq \Gamma(z) \leq \lim_{n\to\infty}\frac{(n)^z(n-1)!}{z(z+1)\cdots(z+n-1)}. \tag{2.4}$$

(2.4) sandwiches $\Gamma(z)$ and forces

$$\forall z \in (0,1], \Gamma(z) = \lim_{n\to\infty}\frac{(n)^z(n-1)!}{z(z+1)\cdots(z+n-1)}. \tag{2.5}$$

Remembering that $\Gamma(z)$ on $\mathbb{R}$ is uniquely defined by $\Gamma(z)$ on $(0,1]$, this completes the proof of the uniqueness of $\Gamma(z)$. ■

Since the above definition only applies on $(0,1]$, we need to extend it to $\mathbb{R}$ in order to find a general definition. Using $\Gamma(z+n) = (\prod_{k=0}^{n-1} z+k)\Gamma(z)$, we find the following definition:

Definition 2.2.

$$\Gamma(z) \equiv \lim_{n\to\infty}\frac{n^z}{z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)^{-1}, \qquad z \notin \mathbb{Z}_{\leq 0}.$$

It will often be easier to use the following equivalent definition:

Definition 2.3 (Euler's Definition)**.**

$$\Gamma(z) \equiv \frac{1}{z}\prod_{n=1}^{\infty}\frac{(1+\frac{1}{n})^z}{1+\frac{z}{n}}, \qquad z \notin \mathbb{Z}_{\leq 0}.$$

Although we only proved this definition for $z \in \mathbb{R}$, it also works for $z \in \mathbb{C}$. The above proof was first developed by Bohr and Mollerup in [2].

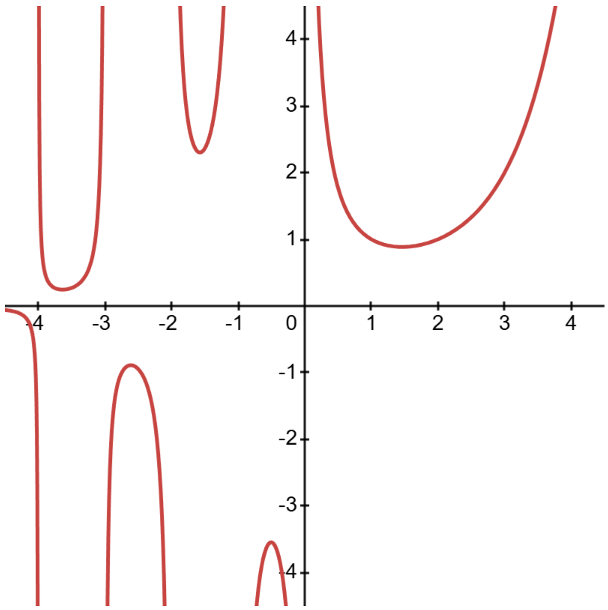

Figure 1. Plot of $\Gamma(z)$

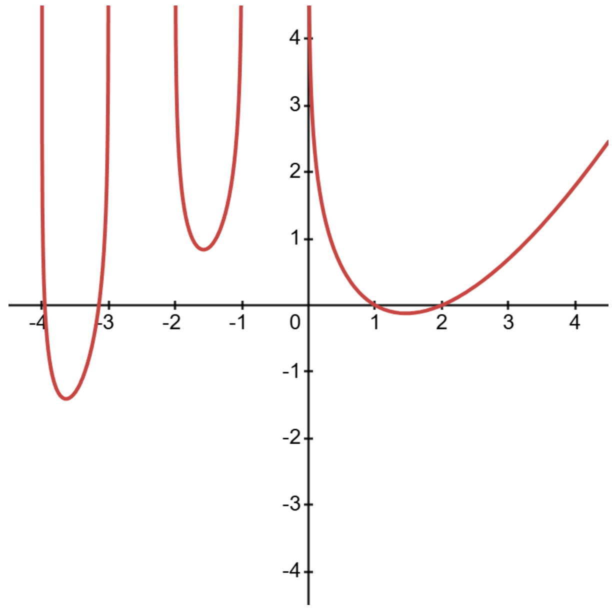

Figure 2. Plot of $\ln\big(\Gamma(z)\big)$

2.2. **Definitions of the Gamma Function.**
There are many other equivalent definitions of the gamma function. Definition 2.3 will be useful for computing the *logarithmic derivative* of $\Gamma(z)$. The Gamma function can also be defined as an improper integral:

Definition 2.4.

$$\Gamma(z) \equiv \int_0^\infty t^{z-1}e^{-t}dt, \qquad \Re(z) > 0$$

The restriction on $\Re(z)$ is necessary for the integral to converge. This definition often appears in physical applications.

Theorem 2.5.

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\frac{(1+\frac{1}{n}^{z}}{1+\frac{z}{n}} = \int_0^\infty t^{z-1}e^{-t}dt, \qquad \Re(z) > 0$$

Proof. Consider the function

$$\Gamma(z,n) = \int_0^n (1-\frac{t}{n}^{n})t^{z-1}dt.$$

By the identity $e^t \equiv \lim_{n\to\infty}(1-\frac{t}{n}^{n}$, we get $\Gamma(z) = \lim_{n\to\infty}\Gamma(z,n)$. Now let $u = t/n$ and apply integration by parts to get

$$\Gamma(z,n) = (1-u)^n\frac{u^z}{z}\bigg|_0^n + \int_0^1 n(1-u)^{n-1}\frac{u^z}{z}du. = \int_0^1 n(1-u)^{n-1}\frac{u^z}{z}du.$$

By repeatedly applying integration by parts, we arrive at

$$\Gamma(z) = \lim_{n\to\infty}\prod_{k=0}^{n}\frac{k+1}{k+z}n^z \tag{2.6}$$

After simplifying, (2.6) is equivalent to Definition2.3, as desired. ■

Another definition of the Gamma function is the Weierstrass definition:

Definition 2.6 (Weierstrass' Definition)**.**

$$\Gamma(z) \equiv \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+\frac{z}{n}}, \qquad z \notin \mathbb{Z}_{\leq 0}$$

Where γ is the *Euler-Mascheroni constant* defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right) \tag{2.7}$$

Theorem 2.7.

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{(1+\frac{1}{n}^{z}}{1+\frac{z}{n}} = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+\frac{z}{n}}, \qquad z \notin \mathbb{Z}_{\leq 0}$$

Proof.
Consider the function

$$f(z,k) = \frac{1}{z} \prod_{n=1}^{k} \frac{(1+\frac{1}{n}^{z}}{1+\frac{z}{n}}.$$

Notice that $\Gamma(z) = \lim_{k\to\infty} \big(f(z,k)\big)$. If we let $H_k = \sum_{n=1}^{k} \left(\frac{1}{n}\right)$ be a Harmonic number, we get

$$f(z,k) = \frac{1}{z} \prod_{n=1}^{k} \frac{\left(1+\frac{1}{n}\right)^{z}}{1+\frac{z}{n}} = \frac{1}{z} \prod_{n=1}^{k} \frac{e^{z\ln\left(1+\frac{1}{n}\right)}}{1+\frac{z}{n}} = \frac{1}{z} \prod_{n=1}^{k} \frac{e^{z(\ln(n+1)-\ln(n))}}{1+\frac{z}{n}} \tag{2.8}$$

$$= \frac{e^{-z(\ln(1)-\ln(2)+\ln(2)-\ln(3)+\cdots+\ln(k))}}{z} \prod_{n=1}^{k} \left(1+\frac{z}{n}\right)^{-1} \tag{2.9}$$

$$= \frac{e^{-z\ln(k)}}{z} \prod_{n=1}^{k} \left(1+\frac{z}{n}\right)^{-1} = \frac{e^{-z\left(\ln(k)-H_k+H_k\right)}}{z} \prod_{n=1}^{k} \left(1+\frac{z}{n}\right)^{-1} \tag{2.10}$$

$$= \frac{e^{-z\left(\ln(k)-H_k\right)}}{z} \prod_{n=1}^{k} \frac{e^{z/n}}{1+\frac{z}{n}} \tag{2.11}$$

Taking the limit of (2.11) as $k \to \infty$, we find

$$\Gamma(z) = \lim_{k\to\infty} f(z,k) = \lim_{k\to\infty} \left(\frac{e^{-z\left(\ln(k)-H_k\right)}}{z} \prod_{n=1}^{k} \frac{e^{z/n}}{1+\frac{z}{n}} \right) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+\frac{z}{n}}$$

Where we have used (2.7) to substitute γ.

∎

2.3. **The Digamma Function.**

When studying the behavior of a function, it's often helpful to study it's derivative. However, calculating the derivative of the $\Gamma(z)$ turns out to be incredibly difficult. Instead, we study the derivative of $\ln\Gamma(z)$, or the *logarithmic derivative* of the $\Gamma(z)$. This is known as the *Digamma Function* and is denoted by $\psi(z)$.

Definition 2.8 (The Digamma Function)**.**

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

Using Definition 2.6, it's somewhat trivial to find a neat series representation for the Digamma function.

Theorem 2.9.

$$\psi(z+1) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right)$$

Proof. Remembering the definition of $\psi(z)$, our first step is to find $\ln(\Gamma(z))$. Using Definition 2.6, we get

$$ln\Gamma(z) = -\gamma z - \ln(z) + \sum_{n=1}^{\infty}\left(\frac{z}{n} - \ln\left(1+\frac{z}{n}\right)\right)$$

Differentiating both sides(and offsetting z by 1), we find that

$$\psi(z+1) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right),$$

as desired. ■

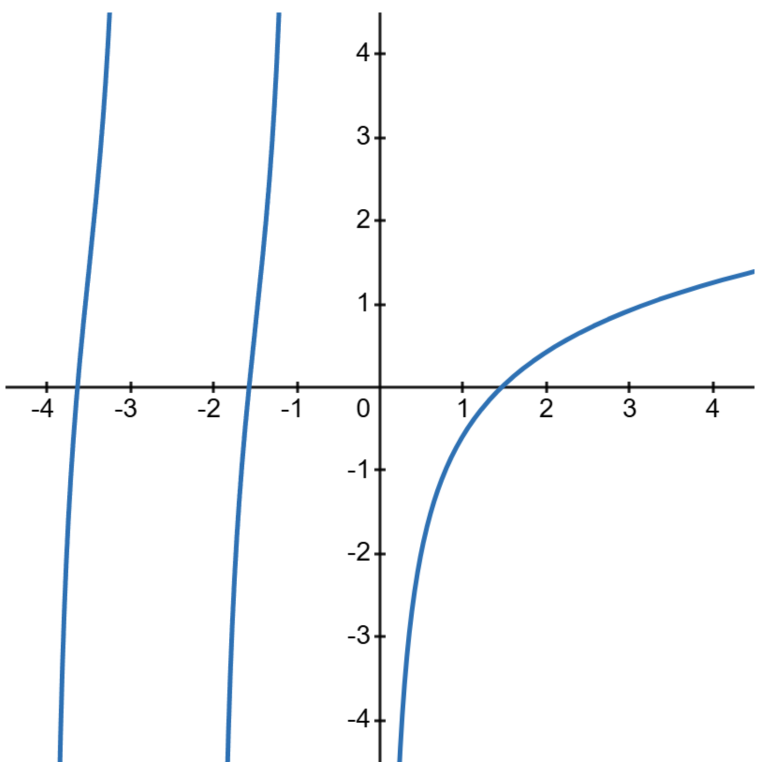

Figure 3. Plot of $\psi(z)$

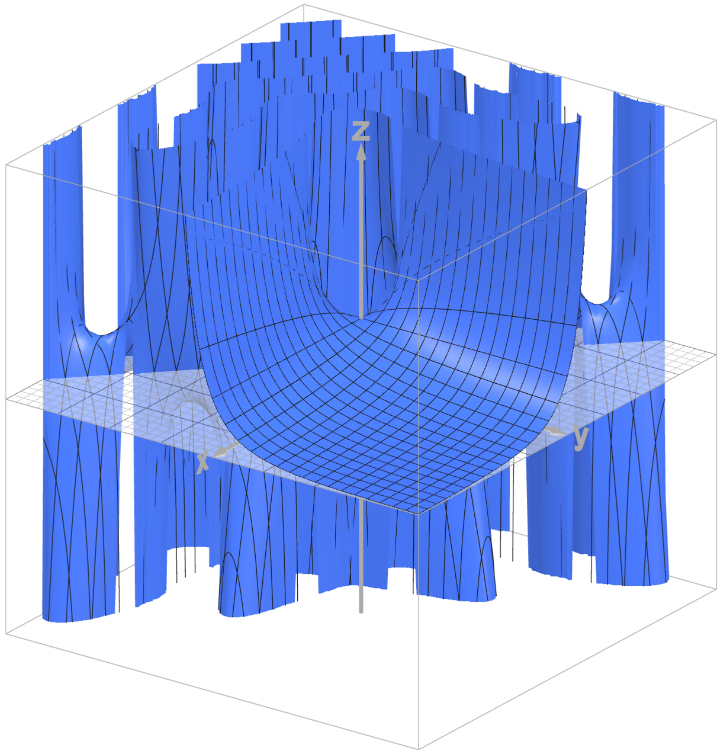

Figure 4. Plot of $B(x,y)$ on $[-4,4] \times [-4,4]$

2.4. **The Beta Function.**

The Beta function plays a fundamental role in statistics. It naturally arises when determining the probability distribution of the probability of an event. If the probability of a success is t and we observe k successes out of n trials, the likelihood of this is given by the expression $t^k(1-t)^{n-k}$. Therefore, the probability that the true likelihood of the event is t is proportional to the above expression. Normalizing the probability, we obtain the following probability distribution:

$$f(t;k+1,n-k+1) = \frac{t^k(1-t)^{n-k}}{\int_0^1 s^k(1-s)^{n-k}ds}$$

From which we define the Beta function using the normalization factor.

Definition 2.10 (The Beta Function)**.**

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

The definition of the Beta function appears similar to that of the Gamma function. Indeed, the two functions are deeply related. The Beta function can be expressed quite elegantly in terms of the Gamma function.

Theorem 2.11.

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. By the definition of the Gamma function,

$$\Gamma(x)\Gamma(y) = \int_0^\infty t^{x-1}e^{-t}dt \int_0^\infty t^{y-1}e^{-s}ds = \int_0^\infty \int_0^\infty t^{x-1}s^{y-1}e^{-t-s}dsdt. \tag{2.12}$$

Now substitute $t = u^2, s = v^2$ to get

$$\Gamma(x)\Gamma(y) = 4\int_0^\infty \int_0^\infty u^{2x-1}v^{2y-1}e^{-u^2-v^2}dsdt. \tag{2.13}$$

Switching to polar coordinates with $u = r\cos(\theta), v = r\sin(\theta)$, (2.13) becomes

$$\Gamma(x)\Gamma(y) = 4\int_0^\infty \int_0^{\frac{\pi}{2}} (r\cos(\theta))^{2x-1}(r\sin(\theta))^{2y-1}e^{-r^2}rd\theta dr \tag{2.14}$$

$$= 4\int_0^\infty r^{2x+2y-1}e^{-r^2}dr \int_0^{\frac{\pi}{2}} \cos(\theta)^{2x-1}\sin(\theta)^{2y-1}d\theta. \tag{2.15}$$

Now we can evaluate each integral separately. For the first integral, make the substitution $z = r^2$ to get

$$\frac{1}{2}\int_0^\infty z^{x+y-1}e^{-z}dz = \frac{1}{2}\Gamma(x+y). \tag{2.16}$$

For the second integral, let $w = \sin(\theta)^2$

$$\frac{1}{2}\int_0^1 (1-w)^{x-1}w^{y-1}dw = \frac{1}{2}B(x,y) \tag{2.17}$$

Substituting (2.16) and(2.17) into (2.15), we find that

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)B(x,y) \implies B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

■

2.5. **Some Integral Identities.**

In the next section, we will use several lemmas concerning specific integrals or families of integrals. I have put the proofs of these lemmas here for convenience.

Lemma 2.12. *Suppose f is a function satisfying the following properties*

(1) $\lim_{t\to 0^+} = f(0)$.
(2) $\int_0^{\frac{\pi}{2}} f(t)dt$ *exists.*
(3) $\int_0^{\frac{\pi}{2}} |f(t)|dt$ *exists.*

Then

$$\lim_{n\to\infty} \frac{\int_0^{\frac{\pi}{2}} f(t)\cos^{2n} t\ dt}{\int_0^{\frac{\pi}{2}} \cos^{2n} t\ dt} = f(0).$$

Proof.
From the Maclaurin series of $\cos(t)$, we see that

$$\int_0^{\frac{\pi}{2}} \cos^{2n} t\ dt \geq \int_0^{\frac{\pi}{2}} \left(1 - \frac{2t}{\pi}\right)^{2n} dt = \frac{\pi}{2(2n+1)}. \tag{2.18}$$

Now let $\epsilon \in \mathbb{R}^+$ be arbitrary and choose $\delta \in \mathbb{R}^+$ such that $\forall t \in (0, \delta),\ |f(t) - f(0)| < \frac{\epsilon}{2}$(The existence of such constants are implied by condition 1.). Then we get

$$\begin{aligned} 0 &\leq \left| \int_0^{\frac{\pi}{2}} f(t)\cos^{2n} t\ dt - \int_0^{\frac{\pi}{2}} f(0)\cos^{2n} t\ dt \right| \leq \int_0^{\frac{\pi}{2}} |f(t) - f(0)| \cos^{2n} t\ dt \\ &= \int_0^{\delta} |f(t) - f(0)|\cos^{2n} t\ dt + \int_{\delta}^{\frac{\pi}{2}} |f(t) - f(0)|\cos^{2n} t\ dt \\ &\leq \frac{\epsilon}{2}\int_0^{\delta} \cos^{2n} t\ dt + \left(\int_{\delta}^{\frac{\pi}{2}} |f(t)|dt + \frac{\pi}{2}|f(0)| \right)\cos^{2n}(\delta) \end{aligned}$$

Dividing by $\int_0^{\frac{\pi}{2}} \cos^{2n} t\ dt$ and using (2.18) gives

$$0 \leq \left| \frac{\int_0^{\frac{\pi}{2}} f(t)\cos^{2n} t\ dt}{\int_0^{\frac{\pi}{2}} \cos^{2n} t\ dt} - f(0) \right| \leq \frac{\epsilon}{2} + Cn\cos^{2n}(\delta). \tag{2.19}$$

Where C is some positive constant. Since we can force $\delta < \frac{\pi}{2}$, the rightmost expression in (2.19) is less than ϵ for large enough n. Since (2.19) is true for any positive ϵ, we can choose arbitrarily small ϵ, thus proving Lemma 2.12. ■

Lemma 2.13 (Wallis' Integral)**.**

$$\int_0^{\frac{\pi}{2}} \cos^{2n}(t)dt = \frac{\pi}{2}\prod_{k=1}^{n} \frac{2k-1}{2k}.$$

Proof. Let $W_{2n} = \int_0^{\frac{\pi}{2}} \cos^{2n}(t)dt$. Then we can write

$$W_{2n} = \int_0^{\frac{\pi}{2}} \cos^{2n}(t)dt = \int_0^{\frac{\pi}{2}} \cos^{2n-2}(t)\big(1 - \sin^2(t)\big)dt \tag{2.20}$$

$$= W_{2(n-1)} - \int_0^{\frac{\pi}{2}} \cos^{2n-2}(t)\sin^2(t)dt. \tag{2.21}$$

Applying integration by parts to the last integral with $dv = \sin(t)\cos^{2n-2}(t)dt$ and $u = \sin(t)$, we find

$$\int_0^{\frac{\pi}{2}} \cos^{2n-2}(t)\sin^2(t)dt = \frac{\cos^{2n-1}(t)}{2n-1}\sin(t)\Big|_0^{\frac{\pi}{2}} + \frac{1}{2n-1}\int_0^{\frac{\pi}{2}} \cos^{2n-1}(t)\cos(t)dt = \frac{W_{2n}}{2n-1}.$$

Substituting this into (2.21), we get

$$W_{2n} = \frac{2n-1}{2n}W_{2n-2}.$$

Using $W_0 = \int_0^{\frac{\pi}{2}} 1\ dt = \frac{\pi}{2}$, we get

$$W_{2n} = \frac{\pi}{2}\prod_{k=1}^{n} \frac{2k-1}{2k}.$$

As desired. ■

Lemma 2.14 (Riemann-Lebesgue Lemma)**.** *Suppose $g(t)$ is a function which satisfies the following properties*

(1) $\int_a^b g(t)dt$ *exists.*

(2) $\int_a^b |g(t)|dt$ *exists.*

Then

$$\lim_{k\to\infty}\int_a^b g(t)\sin(kt)dt = 0.$$

Proof. Start by defining

$$I(k) = \int_a^b g(t)\sin(kt)dt.$$

Notice that we can write

$$I(k) = \int_a^{a+\frac{\pi}{k}} g(t)\sin(kt)dt + \int_{a+\frac{\pi}{k}}^b g(t)\sin(kt)dt \tag{2.22}$$

and

$$I(k) = \int_a^{b-\frac{\pi}{k}} g(t)\sin(kt)dt + \int_{b-\frac{\pi}{k}}^b g(t)\sin(kt)dt. \tag{2.23}$$

Make the substitution $x = t - \frac{\pi}{k}$ in the second integral of (2.22) to get

$$\int_{a+\frac{\pi}{k}}^b g(t)\sin(kt)dt = -\int_a^{b-\frac{\pi}{k}} g(x+\frac{\pi}{k})\sin(kx)dx. \tag{2.24}$$

Substituting this into (2.22), we can write $I(k)$ as a half sum of (2.22) and (2.23) to find

$$I(k) = \frac{1}{2}\int_a^{a+\frac{\pi}{k}} g(t)\sin(kt)dt + \frac{1}{2}\int_{b-\frac{\pi}{k}}^b g(t)\sin(kt)dt + \frac{1}{2}\int_a^{b-\frac{\pi}{k}} \left(g(x) - g(x+\frac{\pi}{k})\right)\sin(kx)dx.$$

As we take the limit of $k\to\infty$, the integrals vanish, producing the desired result. ∎

Lemma 2.15. *Suppose h(t) is a function satisfying the following properties*

(1) $h(t)$ *is differentiable at* $t = s$.

(2) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} h(t)dt$ *exists.*

(3) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |h(t)|dt$ *exists.*

Then

$$\lim_{k\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h(t)}{t-s}\sin(2k(t-s))dt = \begin{cases} \pi h(s), & |s| < \frac{\pi}{2}; \\ \frac{\pi}{2}h(s), & |s| = \frac{\pi}{2}; \\ 0, & |s| > \frac{\pi}{2}. \end{cases}$$

Proof. Proceed by cases.

Case 1. $|s| < \frac{\pi}{2}$**:**

The properties of $h(t)$ imply that both $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h(t)-h(s)}{t-s}dt$ and $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\frac{h(t)-h(s)}{t-s}\right|dt$ exist.

Therefore, we can apply a version of Lemma 2.14 with $g(t) = \frac{h(t)-h(s)}{t-s}$ to get

$$\begin{aligned} 0 &= \lim_{k\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h(t)-h(s)}{t-s}\sin(2k(t-s))dt \\ &= \lim_{k\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h(t)}{t-s}\sin(2k(t-s))dt - h(s)\lim_{k\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(2k(t-s))}{t-s}dt. \end{aligned}$$

Which becomes

$$\lim_{k\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{h(t)}{t-s}\sin(2k(t-s))dt = h(s)\lim_{k\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin(2k(t-s))}{t-s}dt.$$

Let $u = 2k(t-s)$ in the RHS of the equation to get

$$\begin{aligned}\lim_{k\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{h(t)}{t-s}\sin(2k(t-s))dt &= h(s)\lim_{k\to\infty}\int_{-2k(\frac{\pi}{2}+s)}^{2k(\frac{\pi}{2}-s)}\frac{\sin(u)}{u}du \\ &= h(s)\int_{-\infty}^{\infty}\frac{\sin(u)}{u}du \\ &= \pi h(s).\end{aligned}$$

As desired.

Case 2. $|s| = \frac{\pi}{2}$**:**

By the same reasoning as Case 1, we have

$$\begin{aligned}\lim_{k\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{h(t)}{t-\frac{\pi}{2}}\sin(2k(t-\frac{\pi}{2}))dt &= h(\frac{\pi}{2})\lim_{k\to\infty}\int_{-2k(\frac{\pi}{2}+\frac{\pi}{2})}^{2k(\frac{\pi}{2}-\frac{\pi}{2})}\frac{\sin(u)}{u}du \\ &= h(\frac{\pi}{2})\int_{-\infty}^{0}\frac{\sin(u)}{u}du \\ &= \frac{\pi}{2}h(s).\end{aligned}$$

As desired.

Case 3. $|s| > \frac{\pi}{2}$**:**

Since s is no longer in the bounds of integration, the denominator $(t-s)$ is nonzero. Therefore, by the properties of $h(t)$, both $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{h(t)}{t-s}dt$ and $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left|\frac{h(t)}{t-s}\right|dt$ exist. The result now follows directly from Lemma 2.14 with $g(t) = \frac{h(t)}{t-s}$.

■

Lemma 2.16.

$$\int_{-k}^{k}\cos(x)e^{i\theta x}dx = \frac{\sin(k(1-\theta))}{1-\theta} + \frac{\sin(k(1+\theta))}{1+\theta}, \qquad k\in\mathbb{R}^+, \theta\in\mathbb{R}.$$

Proof. We define

$$I = \int_{-k}^{k}\cos(x)e^{i\theta x}dx.$$

Applying integration by parts twice, the expression becomes

$$I = \frac{2\sin(k\theta)\cos(k)}{\theta} - \frac{2\sin(k)cos(\theta k)}{\theta^2} + \frac{I}{\theta^2}.$$

Rearranging for I and applying the identity $2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta)+\sin(\alpha-\beta)$, we get

$$I = \frac{(\theta-1)\sin(k\theta+\theta)+(\theta+1)\sin(k\theta-\theta)}{\theta^2-1} = \frac{\sin(k(1-\theta))}{1-\theta} + \frac{\sin(k(1+\theta))}{1+\theta}.$$

■

3. Continuous Binomial Identities

Using $\Gamma(z)$ to extrapolate the factorial, we can define a generalization of the binomial coefficients on $\mathbb{R}$.

Definition 3.1.

$$\binom{y}{x} = \frac{\Gamma(1+y)}{\Gamma(1+x)\Gamma(1+y-x)}, \qquad y \in \mathbb{R} \backslash \mathbb{Z}_{\leq 0}.$$

Where $\mathbb{Z}_{\leq 0}$ denotes the non-positive integers.

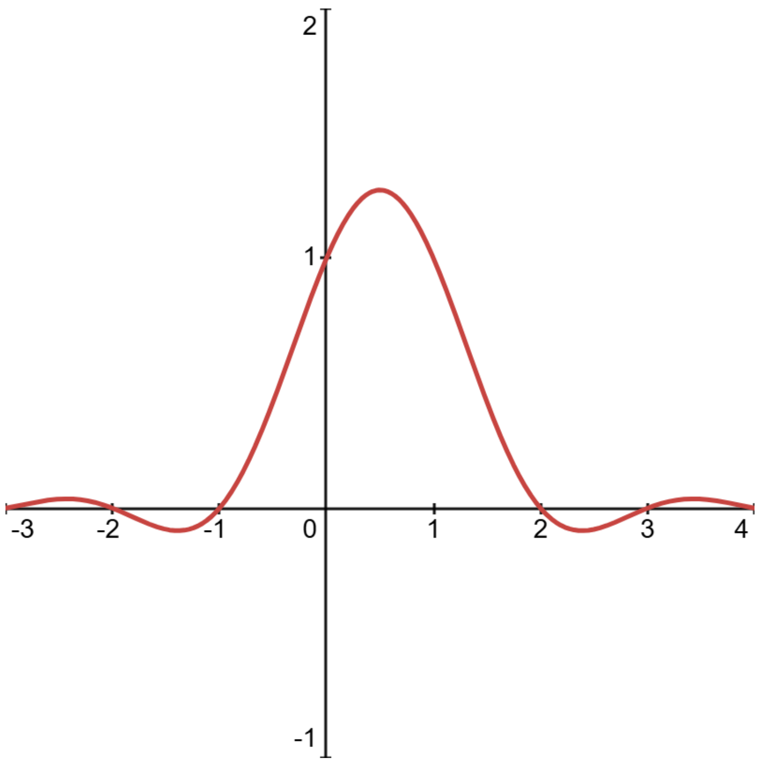

Figure 5. Plot of $\binom{1}{x}$

Remembering Theorem 2.11 Definition 3.1 can be expressed in terms of the Beta function as

$$\binom{y}{x} = \frac{1}{xB(x, 1+y-x)}. \tag{3.1}$$

Proof of (3.1) is left to the reader. Now we need to check the legitimacy of this extension. To do this, let's verify a fundamental property of the binomial coefficients. Namely, a version of the binomial theorem.

Theorem 3.2 (Binomial Theorem)**.**

$$(1+x)^y = \sum_{n=0}^{\infty} \binom{y}{n} x^n, \qquad |x| < 1$$

Proof. Start by writing the Maclaurin series of $(1+x)^y$. For $|x|<1$,

$$(1+x)^y = \sum_{n=0}^{\infty} \frac{y(y-1)\cdots(y-n+1)}{n!} x^n. \tag{3.2}$$

Remembering $\Gamma(z+1) = z\Gamma(z)$, we have $\frac{\Gamma(y+1)}{\Gamma(1+y-n)} = y(y-1)\cdots(y-n+1)$. Using Definition 3.1, (3.2) can be written as

$$(1+x)^y = \sum_{n=0}^{\infty} \binom{y}{n} x^n.$$

∎

Remark 3.3. Theorem 3.2 also holds for $x=1$ if $y>-1$ and for $x=-1$ if $y>0$.

Slightly more general restrictions on x and y are possible in Theorem 3.2. These are outlined in [5]. Notice that when y is an integer, $\forall n > y$, $\binom{y}{n} = 0$, so Theorem 3.2 becomes $(1+x)^y = \sum_{n=0}^{y} \binom{y}{n} x^n$, which is the binomial theorem on $\mathbb{N}$. The restriction of x to $(-1,1)$ is needed for the Taylor series to converge, but for $|x|>1$ we can write

$$(1+x)^y = x^y(1+x^{-1})^y = \sum_{n=0}^{\infty} \binom{y}{n} x^{y-n}.$$

Somewhat surprisingly, this extension of the binomial coefficients has useful applications to statistics, specifically with the negative binomial distribution(This can be used to model the distribution of organisms, as shown in [4].

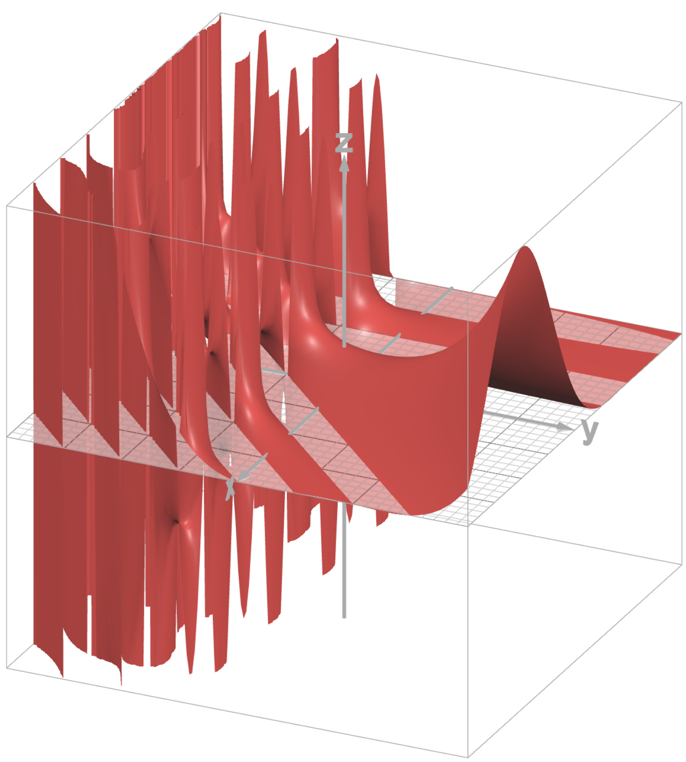

Figure 6. Plot of $\binom{y}{x}$ on $[-4,4]\times[-4,4]$

It's difficult to see the finer details of Figure 6. A deeper explanation of the graph can be found in [3]. Besides the binomial theorem, many other properties of the binomial coefficients are carried over to the continuous case. Since the Maclaurin series of $(1+x)^y$ converges at $x = 1$ for $y > -1$, it follows directly from Theorem 3.2 that

Theorem 3.4.

$$\sum_{k=0}^{\infty} \binom{y}{k} = 2^y, \quad y > -1.$$

Proof of Theorem 3.4 is left to the reader. Furthermore, these properties follow from Definition 3.1:

$$\binom{y}{x} = \binom{y}{y-x}. \tag{3.3}$$

$$\binom{y}{x} = \frac{y}{x}\binom{y-1}{x-1}. \tag{3.4}$$

$$\binom{y}{x} = \binom{y-1}{x-1} + \binom{y-1}{x}, \quad x \neq 0. \tag{3.5}$$

$$\binom{y}{x}\binom{x}{z} = \binom{y}{z}\binom{y-z}{x-z}, \quad z \in \mathbb{Z}, z \leq y. \tag{3.6}$$

Proofs of the above equations can be found in [6]. Knowing that our continuous binomial distribution satisfies many discrete properties of the traditional binomial coefficients, it's natural to ask whether there are continuous analogues to (3.4) and Theorem 3.2. The answer, it turns out, is yes. But in order to prove this, we must first show a integral representation of the continuous binomial coefficients. Using Definition 2.3, we can also express the binomial distribution as an infinite product.

Lemma 3.5.

$$\binom{y}{\frac{y}{2}+x} = \binom{y}{\frac{y}{2}} \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{(k+\frac{y}{2}^2)}\right)$$

Proof. From Definition 2.3, we get

$$\Gamma\big(1+\frac{y}{2}+x\big) = \lim_{n\to\infty} n^{\frac{y}{2}+x} \prod_{k=1}^{n} \left(1 + \frac{\frac{y}{2}+x}{k}\right)^{-1} \tag{3.7}$$

$$= \lim_{n\to\infty} \left(n^{\frac{y}{2}} \prod_{k=1}^{n} \left(1+\frac{y}{2k}\right)^{-1} \cdot n^{x} \prod_{k=1}^{n} \left(1 + \frac{x}{k+\frac{y}{2}}\right)^{-1}\right), \tag{3.8}$$

and

$$\Gamma\big(1+\frac{y}{2}-x\big) = \lim_{n\to\infty} n^{\frac{y}{2}-x} \prod_{k=1}^{n} \left(1 + \frac{\frac{y}{2}-x}{k}\right)^{-1} \tag{3.9}$$

$$= \lim_{n\to\infty} \left[n^{\frac{y}{2}} \prod_{k=1}^{n} \left(1+\frac{y}{2k}\right)^{-1} \cdot n^{-x} \prod_{k=1}^{n} \left(1 - \frac{x}{k+\frac{y}{2}}\right)^{-1}\right]. \tag{3.10}$$

The factorizations in (3.8) and (3.10) can be verified through simple algebra. Multiplying (3.8) and (3.10), we get

$$\begin{aligned}\Gamma\big(1+\frac{y}{2}-x\big)\Gamma\big(1+\frac{y}{2}+x\big) &= \lim_{n\to\infty}\left[\left(n^{\frac{y}{2}}\prod_{k=1}^{n}\left(1+\frac{y}{2k}\right)^{-1}\right)^{2}\prod_{k=1}^{n}\left(1-\frac{x^2}{(k+\frac{y}{2})^2}\right)^{-1}\right]\\ &= \Gamma\big(1+\frac{y}{2}\big)^2\lim_{n\to\infty}\left[\prod_{k=1}^{n}\left(1-\frac{x^2}{(k+\frac{y}{2})^2}\right)^{-1}\right].\end{aligned}$$

Then using Definition 3.1, we find that

$$\binom{y}{\frac{y}{2}+x} = \binom{y}{\frac{y}{2}}\prod_{k=1}^{n}\left(1-\frac{x^2}{(k+\frac{y}{2})^2}\right). \tag{3.11}$$

■

The above proof comes from [5]. Lemma 3.5 implies that the binomial coefficient $\binom{y}{x}$ has zeros for $x \in \mathbb{Z}_{<0}$ and $x \in \mathbb{Z}_{>y}$, which is confirmed by Definition 3.1. Using Lemma 3.5, we can also find an integral representation of the binomial coefficients.

Theorem 3.6.

$$\binom{y}{\frac{y}{2}+x} = \frac{2^y}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^y(t)\cos(2xt)dt,\ \ x,y\in\mathbb{R}, y>-1.$$

Proof.

Case 1. $x \neq 0$**:**

Applying integration by parts twice, we get

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\cos^y(t)\cos(2xt)dt &= \cos^{y+2}(t)\cdot\frac{\sin(2xt)}{2x}\Big|_0^{\frac{\pi}{2}} + \frac{y+2}{2x}\int_0^{\frac{\pi}{2}}\cos^{y+1}(t)\sin(t)\cdot\sin(2xt)dt\\ &= 0+\frac{y+2}{2x}\Bigg(-\cos^{y+1}(t)\sin(t)\cdot\frac{\cos(2xt)}{(2x)}\Big|_0^{\frac{\pi}{2}}\\ &\quad+\frac{1}{2x}\int_0^{\frac{\pi}{2}}\big[(y+1)\cos^y(t)(-\sin^2(t))+\cos^{y+1}(t)\cdot\cos(t)\big]\cos(2xt)dt\Bigg)\\ &= \frac{y+2}{4x^2}\int_0^{\frac{\pi}{2}}\big[(y+1)\cos^y(t)(\cos^2(t)-1)+\cos^{y+2}(t)\big]\cdot\cos(2xt)dt\\ &= \frac{y+2}{4x^2}\int_0^{\frac{\pi}{2}}\big[(y+2)\cos^{y+2}(t)-(y+1)\cos^y(t)\big]\cdot\cos(2xt)dt\\ &= \frac{(y+2)^2}{4x^2}\int_0^{\frac{\pi}{2}}\cos^{y+2}(t)\cos(2xt)dt\\ &\quad-\frac{(y+1)(y+2)}{4x^2}\int_0^{\frac{\pi}{2}}\cos^y(t)\cos(2xt)dt.\end{aligned}$$

Rearranging gives the following recursive formula:

$$\int_0^{\frac{\pi}{2}}\cos^y(t)\cos(2xt)dt = \frac{y+2}{y+1}\left(1-\frac{x^2}{\big(1+\frac{y}{2}\big)^2}\right)\int_0^{\frac{\pi}{2}}\cos^{y+2}(t)\cos(2xt)dt. \tag{3.12}$$

Case 2. $x = 0$**:**
Similarly to Case 1, we use integration by parts(with $u = \cos^{y-1}(t)$ and $dv = cos(t)dt$) to get

$$\int_0^{\frac{\pi}{2}} \cos^y(t)\cos(0)dt = \cos^{y-1}(t)\sin(t)\Big|_0^{\frac{\pi}{2}} + (y-1)\int_0^{\frac{\pi}{2}} \cos^{y-2}(t)(1-\cos^2(t))dt$$
$$= (y-1)\int_0^{\frac{\pi}{2}} \cos^{y-2}(t)dt - (y-1)\int_0^{\frac{\pi}{2}} \cos^y dt.$$

After rearranging, we find

$$\int_0^{\frac{\pi}{2}} \cos^y dt. = \frac{y-1}{y}\int_0^{\frac{\pi}{2}} \cos^{y-2} dt.$$

This is equivalent to (3.12) at $x = 0$.

The above analysis implies that (3.12) holds for any x. Therefore, we can apply (3.12) n times to get

$$\int_0^{\frac{\pi}{2}} \cos^y(t)\cos(2xt)dt = \prod_{k=1}^{n} \frac{y+2k}{y+2k-1} \prod_{k=1}^{n}\left(1-\frac{x^2}{\left(1+\frac{y}{2}\right)^2}\right)\int_0^{\frac{\pi}{2}} \cos^{y+2n}(t)\cos(2xt)dt. \tag{3.13}$$

At $y = x = 0$, the RHS is proportional to Wallis' integral$\left(\int_0^{\frac{\pi}{2}} \cos^{2n}(t)dt\right)$. From Lemma 2.13, we have $\int_0^{\frac{\pi}{2}} \cos^{2n}(t)dt = \frac{\pi}{2}\prod_{k=1}^{n}\frac{2k-1}{2k}$. Or, equivalently,

$$\frac{\frac{\pi}{2}\prod_{k=1}^{n}\frac{2k-1}{2k}}{\int_0^{\frac{\pi}{2}} \cos^{2n}(t)dt} = 1. \tag{3.14}$$

Therefore we can multiply the RHS of (3.13) by the LHS of (3.14) to get

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^y(t)\cos(2xt)dt =& \frac{\pi}{2}\prod_{k=1}^{n}\left(\frac{y+2k}{y+2k-1}\cdot\frac{2k-1}{2k}\right)\prod_{k=1}^{n}\left(1-\frac{x^2}{\left(1+\frac{y}{2}\right)^2}\right)\\ &\cdot \frac{\int_0^{\frac{\pi}{2}} \cos^{2n}(t)\cos^y(t)\cos(2xt)dt}{\int_0^{\frac{\pi}{2}} \cos^{2n}(t)dt}.\end{aligned} \tag{3.15}$$

Now observe that

$$\prod_{k=1}^{n}\left(\frac{y+2k}{y+2k-1}\cdot\frac{2k-1}{2k}\right)$$
$$= \prod_{k=1}^{n}\frac{\frac{y}{2k}+1}{\frac{y}{2k-1}+1} = \frac{\prod_{k=1}^{n}\left(\frac{y}{2k}+1\right)^2}{\prod_{k=1}^{2n}\frac{y}{k}+1} \tag{3.16}$$
$$= \frac{1}{2^y}\cdot\frac{(2n)^y\prod_{k=1}^{2n}\left(\frac{y}{k}+1\right)^{-1}}{\left(n^{y/2}\prod_{k=1}^{n}\left(\frac{y}{2k}+1\right)^{-1}\right)^2}. \tag{3.17}$$

Notice that denominator of the LHS of (3.16) is a product over all odd positive integers, which is equivalent to the ratio of the same product over all positive integers to the product over even integers. (3.16) follows directly from this fact. Now we can take the limit of

(3.15) as $n \to \infty$. 2.2 implies that (3.16) approaches $\frac{\Gamma(y+1)}{2^y\Gamma\left(\frac{y}{2}+1\right)^2} = \frac{1}{2^y}\binom{y}{\frac{y}{2}}$. Since $f(t) = \cos^y(t)\cos(2xt)$ satisfies the conditions of Lemma 2.12, $\frac{\int_0^{\frac{\pi}{2}} \cos^{2n}(t)\cos^y(t)\cos(2xt)dt}{\int_0^{\frac{\pi}{2}} \cos^{2n}(t)dt}$ must approach $f(0) = 1$. Therefore taking the limit of (3.15) and applying Theorem 3.5 yields

$$\int_0^{\frac{\pi}{2}} \cos^y(t)\cos(2xt)dt = \frac{\pi}{2}\frac{1}{2^y}\binom{y}{\frac{y}{2}}\prod_{k=1}^{n}\left(1 - \frac{x^2}{\left(1+\frac{y}{2}\right)^2}\right)$$
$$= \frac{\pi}{2}\frac{1}{2^y}\binom{y}{\frac{y}{2}+x}.$$

Rearranging gives

$$\binom{y}{\frac{y}{2}+x} = \frac{2^y}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^y(t)\cos(2xt)dt.$$

As desired. ■

Remark 3.7. Remembering (3.1), 3.6 can be expressed in terms of the Beta function as

$$\int_0^{\frac{\pi}{2}} \cos^{x+y-2}(t)\cos((y-x)(t))dt = \frac{\pi}{(x+y-1)2^{x+y-1}B(x,y)}.$$

Using Theorem 3.6, we can find a power series representation of the choose function.

Theorem 3.8.

$$\binom{y}{\frac{y}{2}+x} = \frac{2^y}{\pi}\sum_{n=0}^{\infty}(-1)^n a_{2n}x^{2n}.$$

Where we define

$$a_{2n} = \frac{2^{2n}}{(2n)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^{2n}\cos^y(t)dt.$$

Proof. We substitute the Maclaurin series of $\cos(2xt)$ in Theorem 3.6 to get

$$\binom{y}{\frac{y}{2}+x} = \frac{2^y}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^y(t)\sum_{n=0}^{\infty}\frac{(-1)^n(2xt)^{2n}}{(2n)!}dt = \frac{2^y}{\pi}\sum_{n=0}^{\infty}\left[(-1)^n\frac{2^{2n}}{(2n)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^{2n}\cos^y(t)dt \cdot x^{2n}\right].$$

■

We're now able to prove a continuous analogue of Theorem 3.4.

Theorem 3.9.

$$\int_{-\infty}^{\infty}\binom{y}{x}dx = 2^y, \quad y > -1.$$

Proof. Proof of the convergence of this integral for $y > -1$ can be found in [5]. Consider the function

$$f(k) = \int_{\frac{y}{2}-k}^{\frac{y}{2}+k}\binom{y}{x}dx = \int_{-k}^{k}\binom{y}{\frac{y}{2}+x}dx = \int_{-k}^{k}\frac{2^y}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^y(t)\cos(2xt)dt\ dx$$
$$= \frac{2^y}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^y(t)\int_{-k}^{k}\cos(2xt)dx\ dt = \frac{2^y}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos^y(t)}{t}\sin(2kt)dt.$$

The swap of integrals is justified by Fubini's Theorem([1]). Therefore we can write

$$\int_{-\infty}^{\infty} \binom{y}{x} dx = \lim_{k\to\infty} f(k) = \lim_{k\to\infty} \frac{2^y}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^y(t)}{t} \sin(2kt)dt.$$

Since $h(t) = \cos^y(t)$ satisfies the conditions of Lemma 2.15(with $s = 0$), we have

$$\int_{-\infty}^{\infty} \binom{y}{x} dx = \frac{2^y}{\pi}\pi \cos^y(0) = 2^y.$$

As desired. ■

We're also able to show a continuous analogue to the Binomial Theorem. To do so, we must extend it to the complex plane. Although we've only been considering real numbers, all theorems we've proven thus far also hold for complex numbers.

Theorem 3.10 (Continuous Binomial Theorem)**.** *For $y \in \mathbb{R}$, $z \in \mathbb{C}$, we have*

$$\int_{-\infty}^{\infty} \binom{y}{x} z^x dx = (1+z)^y, \qquad |z| = 1,\ y > 0.$$

A slightly more general restriction on y is possible([5]).

Proof. Let $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Then Theorem 3.10 becomes

$$\int_{-\infty}^{\infty} \binom{y}{x} e^{i\theta x} dx = (1+e^{i\theta})^y.$$

Therefore we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \binom{y}{x} e^{i\theta x} dx &= e^{\frac{i\theta y}{2}} \int_{-\infty}^{\infty} \binom{y}{\frac{y}{2}+x} e^{i\theta x} dx = \frac{2^y}{\pi} e^{\frac{i\theta y}{2}} \int_{-\infty}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^y(t)\cos(2xt)dt\ e^{i\theta x}dx \\
&= \frac{2^y}{\pi} e^{\frac{i\theta y}{2}} \int_{-\infty}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^y(t)\cos(2xt)e^{i\theta x}dtdx \\
&= \frac{2^y}{\pi} e^{\frac{i\theta y}{2}} \lim_{k\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^y(t) \int_{-k}^{k} \cos(2xt)e^{i\theta x}dxdt \\
&= \frac{2^y}{\pi} e^{\frac{i\theta y}{2}} \lim_{k\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^y(t) \left[\frac{\sin(2k\big(t-\frac{\theta}{2}\big))}{2(t-\frac{\theta}{2})} + \frac{\sin(2k\big(t+\frac{\theta}{2}\big))}{2(t+\frac{\theta}{2})} \right] dt \\
&= \frac{2^{y-1}}{\pi} e^{\frac{i\theta y}{2}} \lim_{k\to\infty} \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^y t \frac{\sin(2k(t-\frac{\theta}{2}))}{t-\frac{\theta}{2}} dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^y t \frac{\sin(2k(t+\frac{\theta}{2}))}{t+\frac{\theta}{2}} dt \right] \\
&= \frac{2^y}{\pi} e^{\frac{i\theta y}{2}} \lim_{k\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^y(t)}{t-\frac{\theta}{2}} \sin(2k\big(t-\theta/2\big))dt.
\end{aligned}$$

Where we have used Theorem 3.6 to substitute for $\binom{y}{\frac{y}{2}+x}$ and Lemma 2.16 to evaluate the nested integral. Since $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^y(t)dt$ and $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos^y(t)|dt$ both exist, and $\cos^y(t)$ is differentiable, we can use Lemma 2.15 with $h(t) = \cos^y(t)$ and $s = \frac{\theta}{2}$. By the period of $e^{i\theta}$, we can assume $|\theta| \le \pi$.

$$\int_{-\infty}^{\infty} \binom{y}{x} e^{i\theta x} dx = \begin{cases} 2^y e^{i\theta y/2} \cos^y\left(\frac{\theta}{2}\right), & |\theta| < \pi; \\ 2^{y-1} e^{i\theta y/2} \cos^y\left(\frac{\theta}{2}\right), & |\theta| = \pi. \end{cases} \tag{3.18}$$

Case 1. $|\theta| = \pi$**:**
(3.18) gives $2^{y-1}e^0\cos^y\big(0\big) = 0 = (1-1)^y = (1+e^{i\theta})^y$, as desired.
Case 2. $|\theta| < \pi$**:**
(3.18) gives $2^y e^{i\frac{\theta y}{2}}\cos^y\left(\frac{\theta}{2}\right) = \left(2e^{i\frac{\theta}{2}}\cos\left(\frac{\theta}{2}\right)\right)^y = \left(2\cos^2\left(\frac{\theta}{2}\right) + 2i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)^y$.
Simplifying with trigonometric identities, we have $\int_{-\infty}^{\infty}\binom{y}{x}e^{i\theta x}dx = \big(1+e^{i\theta}\big)^y$, as desired.

∎

Remark 3.11. As a special case of Theorem 3.10, we also get the row integration identity for continuous binomial coefficients:

$$\int_{-\infty}^{\infty}\binom{y}{x}dx = 2^y, \quad y > -1. \tag{3.19}$$

4. Investigating $\binom{y}{x}$ As a Function of x

In this section we will use the notation $b_y(x) = \binom{\frac{y}{2}}{\frac{y}{2}+x}$(Or simply $b(x)$) to emphasize that we are only considering $\binom{y}{x}$ as a function of x. We will start by forming a differential equation satisfied by $b_y(x)$:

Theorem 4.1.

$$b'(x) = \left(\psi(\frac{y}{2} - x + 1) - \psi(\frac{y}{2} + x + 1)\right)b(x).$$

Where $\psi(z)$ is the Digamma function defined as the logarithmic derivative of $\Gamma(z)$ in Definition 2.8.

Proof. Remembering Definition 3.1, we have

$$b(x) = \frac{\Gamma(y+1)}{\Gamma(\frac{y}{2} - x + 1)\Gamma(\frac{y}{2} + x + 1)} = \Gamma(y+1)\left[\Gamma\big(\frac{y}{2} - x + 1\big)^{-1}\Gamma\big(\frac{y}{2} + x + 1\big)^{-1}\right].$$

Using the product rule to calculate the derivative, we obtain

$$b'(x) = \frac{\Gamma(y+1}{\Gamma(\frac{y}{2} - x + 1)\Gamma(\frac{y}{2} + x + 1)}\left[\frac{\Gamma'(\frac{y}{2} - x + 1)}{\Gamma(\frac{y}{2} - x + 1)} - \frac{\Gamma'(\frac{y}{2} + x + 1)}{\Gamma(\frac{y}{2} + x + 1)}\right].$$

Substituting using Definition 2.8 and Definition 3.1 yields

$$b'(x) = \left(\psi(\frac{y}{2} - x + 1) - \psi(\frac{y}{2} + x + 1)\right)b(x).$$

As desired. ∎

Remark 4.2. Theorem 4.1 can be restated using the series representation of $\psi(z)$ developed in Theorem 2.9:

$$b'(x) = b(x)\sum_{n=1}^{\infty}\frac{8x}{4x^2 - (2n+y)^2}.$$

An equivalent expression is developed without the use of the Gamma function in [6].

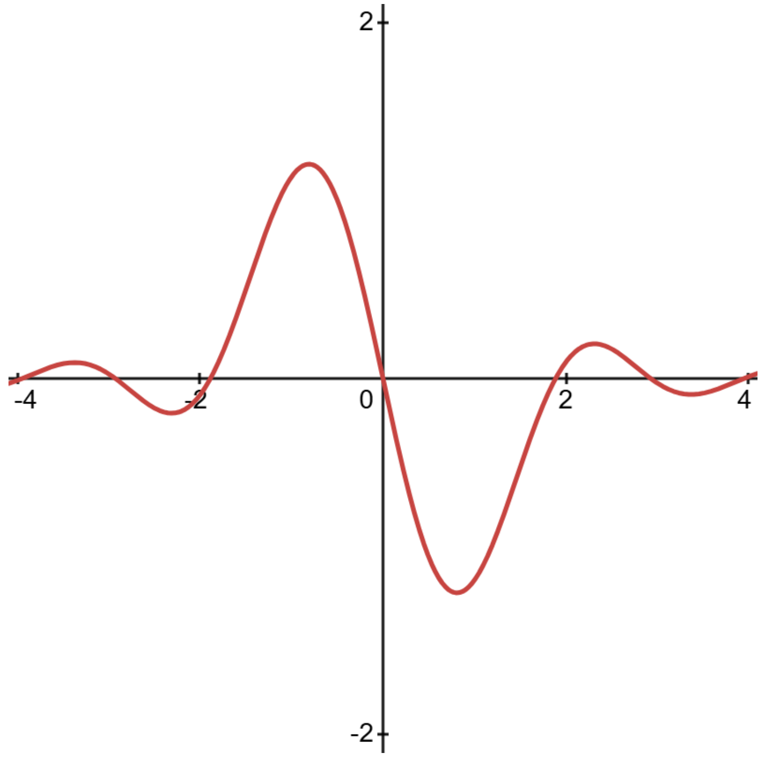

Figure 7. Plot of $b_1'(x)$

Remark 4.3. For $y \in \mathbb{Z}^+$, Theorem 4.1 is equivalent to

$$b'(x) = \left(\pi \cot(\pi x) - \sum_{n=0}^{y} \frac{1}{x-n}\right) b(x).$$

Theorem 4.1 also allows us to find expressions involving $b''(x)$ using the product rule.

Theorem 4.4.

$$b''(x) = \left[\left(\psi(\frac{y}{2} - x + 1) - \psi(\frac{y}{2} + x + 1)\right)^2 - \left(\psi'(\frac{y}{2} - x + 1) + \psi'(\frac{y}{2} + x + 1)\right)\right] b(x).$$

Proof. By Theorem 4.1, we have

$$b''(x) = \frac{d}{dx}\left[\left(\psi(\frac{y}{2} - x + 1) - \psi(\frac{y}{2} + x + 1)\right) b(x)\right].$$

Using the product rule, we get

$$b''(x) = \left(\psi(\frac{y}{2} - x + 1) - \psi(\frac{y}{2} + x + 1)\right) b'(x) - \left(\psi'(\frac{y}{2} - x + 1) + \psi'(\frac{y}{2} + x + 1)\right) b(x).$$

Substituting for $b'(x)$ using Theorem 4.1 gives

$$b''(x) = \left[\left(\psi(\frac{y}{2} - x + 1) - \psi(\frac{y}{2} + x + 1)\right)^2 - \left(\psi'(\frac{y}{2} - x + 1) + \psi'(\frac{y}{2} + x + 1)\right)\right] b(x).$$

■

The same methods as the above proof allow us to calculate higher order derivatives of $b(x)$ to get an infinite family of differential equations satisfied by the binomial coefficients which are displayed below. For simplicity, we have made the substitution $f(x) = \psi(\frac{y}{2} - x + 1) - \psi(\frac{y}{2} + x + 1)$.

$$\begin{aligned}
b^{(1)}(x) &= f(x)b(x). \\
b^{(2)}(x) &= \left[f^2(x) + f^{(1)}(x)\right]b(x). \\
b^{(3)}(x) &= \left[f^3(x) + 3f(x)f^{(1)}(x) + f^{(2)}(x)\right]b(x). \\
b^{(4)}(x) &= \left[f^4(x) + 6f^2(x)f^{(1)}(x) + 4f(x)f^{(2)}(x) + 3f^{(1)}(x)f^{(1)}(x) + f^{(3)}(x)\right]b(x). \\
&\ \vdots
\end{aligned}$$

Equivalent expressions involving the derivatives of $b(x)$ are used in [6] to find the minima and maxima of $b(x)$.

5. Future Work

There is very little work done on studying the continuous binomial coefficients. Many of the identities(such as the continuous binomial theorem) we have proven involve tedious calculations. These identities may have simpler proofs which we have not yet found. Additionally, it may be possible to find a general expression for higher order derivatives of the binomial coefficients using the differential equations proven here.

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