An Introduction to Hyperbolic Geometry

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Euler Circle

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Euclid's Five Postulates

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1. A straight line can be drawn from any point to any other point.
2. A finite straight line can be extended indefinitely.
3. A circle can be drawn with any center and any radius.
4. All right angles are equal to each other.
5. If a straight line that cuts two other lines forms interior angles on the same side whose sum is less than two right angles, then the two lines intersect on that side.

The Controversial Nature of the Fifth Postulate

Unlike the others, the fifth postulate refers to the global behavior of lines at infinity, not to a direct construction. It was considered too complex and unconvincing to be taken as a fundamental axiom. Thus, several mathematicians attempted to prove it as a theorem from the other postulates.

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- None of these efforts succeeded; all implicitly used Euclidean assumptions.
- Eventually, it seemed that the fifth postulate was **logically independent** from the others.

Preserved Logical Structure

In the 19th century, Nikolai Lobachevsky (1829) and János Bolyai (1832) independently developed a non Euclidean geometry (later called hyperbolic geometry) in which the first four postulates hold and the fifth is replaced by the following axiom.

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Hyperbolic Axiom

Given a line l and a point P not on l, there exists at least two lines through P that do not intersect l.

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Models of Hyperbolic Geometry

Is Euclidean geometry consistent? We can prove that hyperbolic geometry is as consistent as Euclidean geometry if we find a **model**. A model represents hyperbolic points, lines, and distances using Euclidean objects.

Open and Closed Sets; Boundary

Definition

Let $A \subset \mathbb{C}$. We say that:

- A is **open** if for every $z \in A$, there exists $\varepsilon > 0$ such that

$$D(z, \varepsilon) = \{w \in \mathbb{C} \mid |w - z| < \varepsilon\} \subset A.$$

- A is **closed** if $\mathbb{C} \setminus A$ is open.
- The **boundary** of A, denoted ∂A, is the set of all $z \in \mathbb{C}$ such that

$$\forall \varepsilon > 0, \quad D(z, \varepsilon) \cap A \neq \emptyset \quad \text{and} \quad D(z, \varepsilon) \cap (\mathbb{C} \setminus A) \neq \emptyset.$$

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Example

- $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ is open, and $\partial \mathbb{H} = \mathbb{R}$.
- The closed disc $\{z \in \mathbb{C} \mid |z| \leq 1\}$ is closed, and its boundary is $\{z \in \mathbb{C} \mid |z| = 1\}$.

Metric and Isometries

Definition

Let M be a set. A function $d : M \times M \to \mathbb{R}$ is called a **metric** if for all $p, q, r \in M$:

- $d(p, q) \geq 0$, and $d(p, q) = 0$ if and only if $p = q$,
- $d(p, q) = d(q, p)$,
- $d(p, r) \leq d(p, q) + d(q, r)$ (triangle inequality).

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Definition

An **isometry** is a map $f : M \to M$ such that for all $p, q \in M$,

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Properties of Isometries

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Let $f : M \to M$ be an isometry on a space equipped with a metric and an area measure defined from it.

- f preserves distances: $d(f(p), f(q)) = d(p, q)$.
- f is conformal: it preserves angles and orientation.
- f preserves geodesics.
- f preserves area (when the area is defined from the metric).

Möbius Transformations

Definition

A **Möbius transformation** is a function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$f(z) = \frac{az + b}{cz + d}, \quad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0.$$

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Proposition

Möbius transformations are conformal on their domain of definition.

Example

- $z \mapsto z + b$: translation,
- $z \mapsto kz$, with $k > 0$: dilation,
- $z \mapsto \frac{1}{z}$: inversion in the unit circle,
- $z \mapsto z$: identity.

Upper Half-Plane Model

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- Points on the extended real line $\mathbb{R} \cup \{\infty\}$ are called **ideal points**; they represent points at infinity.

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- Geodesics in $\mathbb{D}$ are:
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  - arcs of circles orthogonal (i.e., intersecting the boundary circle at right angles in the Euclidean sense) to the boundary circle.
- The boundary $\partial\mathbb{D}$ is called the **ideal boundary**, and represents points at infinity.

Area of a Hyperbolic Triangle

Theorem

For a triangle with angles α, β, γ:

$$\mathit{Area} = \pi - (\alpha + \beta + \gamma)$$

Area in the Upper Half-Plane Model

Proposition

The area of a region $R \subset \mathbb{H}$ is given by

$$Area(R) = \iint_R \frac{dx\,dy}{y^2}.$$

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- This is the formula used to compute hyperbolic area in the upper half-plane.
- It gives finite area even for regions that are unbounded in the Euclidean sense.
- The formula does not change under transformations that preserve the upper half-plane.

Lemma: Triangle with Ideal Vertex

Lemma

Let $T \subset \mathbb{H}$ be a hyperbolic triangle with one ideal vertex. If the interior angles at the other two vertices are α and β, then

$$Area(T) = \pi - (\alpha + \beta).$$

Sketch of Proof of the Area Formula

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Using the area formula:

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But in the original triangle all three angles are present, so we obtain

$$\text{Area}(T) = \pi - (\alpha + \beta + \gamma).$$

Thanks

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