AN INTRODUCTION TO HYPERBOLIC GEOMETRY

ROY YARANGA

Abstract. This paper presents a basic introduction to hyperbolic geometry and proves the formula for the area of a triangle in the hyperbolic plane. The approach tries to avoid unnecessary generalizations and focuses only on the essential concepts.

1. Introduction

Hyperbolic geometry is an alternative to Euclidean geometry that arises by replacing the parallel postulate. In this geometry, given a point outside a line, there exist infinitely many lines through the point that do not intersect the given line. This leads to a coherent system with properties fundamentally different from those of Euclidean geometry.

The main goal of this paper is to prove the formula for the area of a triangle in the hyperbolic plane. If a triangle has interior angles α, β, and γ, then its area is

$$\text{Area} = \pi - (\alpha + \beta + \gamma).$$

This paper aims to present the proof in the simplest possible terms, avoiding unnecessary generalizations and focusing on the fundamental concepts needed to understand the result.

To establish this formula, we proceed as follows.

In Section 2, we review Euclid's parallel postulate and present the alternative hypothesis that leads to hyperbolic geometry.

In Section 3, we introduce the fundamental axioms of hyperbolic geometry and study some of their basic consequences, including the existence of parallel lines and characteristic angular properties.

In Section 4, we describe the classical models of the hyperbolic plane, which allow us to represent its points and lines within the Euclidean plane.

In Section 5, we define the necessary metric notions, such as distance and area, and finally derive the formula for the area of a triangle.

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2. The fifth postulate

Around 300 BC, Euclid wrote one of the most influential works in mathematics. *The Elements* is a 13-volume treatise in which Euclid compiled a vast number of the major mathematical discoveries known up to that time, especially in geometry and number theory. In these books, he proposed a new approach in which every mathematical result can be derived through logical reasoning from sets of propositions that are asserted without any proof, called axioms or postulates. In his first book, Euclid proposed five postulates for geometry.

(1) Let it have been postulated to draw a straight line from any point to any point.
(2) And to produce a finite straight line continuously in a straight line.
(3) And to draw a circle with any center and radius.
(4) And that all right angles are equal to one another.
(5) And that if a straight line falling across two other straight lines makes internal angles on the same side of itself whose sum is less than two right angles, then the two other straight lines, being produced to infinity, meet on that side of the original straight-line that the sum of the internal angles is less than two right angles and do not meet on the other side.

We can see that the fifth postulate is too long compared to the others. We will occasionally refer to this as the Parallel Postulate because it is equivalent to the following proposition:

> Given a line and a point not on the line, there is at most one line through the point that is parallel to the given line.

This way of stating the fifth postulate is also known as Playfair's axiom because mathematician John Playfair referenced it in his book *Elements of Geometry*. The first four postulates were accepted without objection, however, the fifth was controversial. The way the fifth postulate was originally stated was too long compared to the others. Axioms are supposed to be as simple and elementary as possible, right? This prompted mathematicians to question whether the fifth postulate was really necessary, that is, whether it was actually a theorem that could be proved from the first four.

2.1. **Attempts to prove the fifth Postulate:**

For approximately two millennia, many mathematicians attempted to prove Euclid's fifth postulate. One of the first to point this out was Proclus (500 AD) in his *Commentary on the first book of Euclid's Elements*. His reasoning was based on the fact that if a transversal cuts two lines forming congruent alternate interior angles, then the lines will not meet when extended in the corresponding direction and must therefore be considered parallel. This assertion is based, however, on an unjustified assumption: that parallel lines are equidistant. This property cannot be deduced without the parallel postulate, and in fact, it does not hold in non-Euclidean geometries such as hyperbolic geometry.

John Wallis (1616–1703), one of England's leading mathematicians of the 17th century, approached the problem of the fifth postulate from a different perspective. Instead of directly attacking the postulate, he proposed as an axiom a geometric property that he considered more intuitive: the possibility of constructing similar triangles of any size while maintaining constant the ratio of their sides to their angles. In particular, he argued that for any given triangle, it is possible to construct an arbitrarily large one with the same angles, which is today called the principle of angular similarity.

This principle implies, however, the postulate of parallels. Indeed, the existence of similar triangles cannot be guaranteed in the absence of the postulate of parallels. We will see that in non-Euclidean geometries, such as hyperbolic geometry, triangles do not simultaneously preserve angles and side ratios; the notion of similarity does not behave in a Euclidean manner. Therefore, by using the principle of similarity as an axiom, Wallis was engaging in circular reasoning.

The Italian mathematician Giovanni Girolamo Saccheri (1667-1733) published the book called *Euclid Freed of Every Flaw*. His idea of proof was using contradiction.

Saccheri studied quadrilaterals whose base angles were right angles and whose sides adjacent to the base were congruent, which we will call Saccheri quadrilaterals.

Saccheri proved that the summit angles were congruent using Euclid's first four axioms, from which he derived three possible cases:

- Summit angles are right angles.
- Summit angles are obtuse.
- Summit angles are acute.

His idea was to prove that the last two cases led to a contradiction. Saccheri successfully proved that the second was contradictory, but he never found a contradiction in the third. In fact, there seemed to be some consistency in this hypothesis, although Saccheri simply dismissed it because he believed that these ideas went against Euclidean intuition.

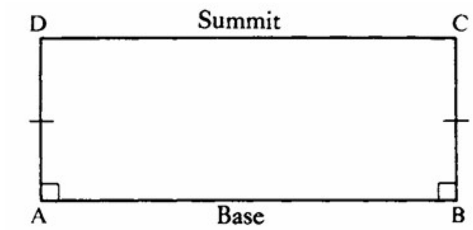

Figure 1. Saccheri quadrilaterals.

Adrien-Marie Legendre (1752-1833) devoted a significant portion of his mathematical work to clarifying the foundations of geometry. He rejected the logical independence of the parallel postulate and attempted to prove that the sum of the interior angles of any triangle is equal to two right angles, without assuming the parallel postulate. He believed that this statement was more intuitively obvious and thus more suitable as a foundation for geometry.

To support this claim, he constructed sequences of triangles sharing a common base and with vertices increasingly distant from that base, hoping to show that any angular deviation would vanish as the triangles grew. He reasoned that, if the angle sum were not always constant, it would lead to contradictions in the structure of geometry.

However, his argument inevitably relied on assumptions that are logically equivalent to the parallel postulate. In particular, the claim that all triangles have the same angle sum implies the Euclidean version of parallelism. In hyperbolic geometry, for instance, triangles have an angle sum strictly less than two right angles. Thus, Legendre's proof presupposes the very conclusion it aims to establish.

There have been many other historical attempts to prove the parallel postulate, ranging from naive constructions to more sophisticated arguments. A detailed account of these efforts can be found in [Bon55].

3. Axioms of hyperbolic geometry

Hyperbolic geometry is the deductive system obtained by preserving Euclid's first four postulates and replacing the fifth with a statement incompatible with it. The notions of points, lines, segments, and angles remain unchanged, as does the validity of the basic constructions. This framework allows for the development of a coherent geometry in which all provable propositions are maintained without resorting to the parallel postulate.

The statement that replaces the fifth postulate is:

> At least two distinct lines that do not intersect a given line pass through a point outside a given line.

This new formulation does not require the rejection of any other principle accepted in the Elements. From now on, all consequences will be derived solely from the unchanged postulates and this new version of parallelism.

3.1. **Basic properties of hyperbolic geometry:**

Although it shares its syntactical foundations with Euclidean geometry, hyperbolic geometry differs notably in its metric and structural consequences. These differences do not contradict the preserved postulates, but necessarily follow from the new conception of parallelism. We will present some notable properties.

Proposition 3.1. *Rectangles don't exist.*

Proof. (sketch) This statement is equivalent to the parallel postulate. ■

Corollary 3.2. *The sum of the interior angles of a triangle is strictly less than two rigth angles.*

Corollary 3.3. *The sum of the interior angles of any convex quadrilateral is strictly less than* $360°$.

Proposition 3.4. *Every similar triangles are congruent in hyperbolic geometry.*

Proof. (sketch)

Assume there exist two similar but noncongruent triangles. Without loss of generality, suppose one triangle is larger than the other.

Construct points on the larger triangle to match two sides of the smaller one. By SAS, the smaller triangle is congruent to the subtriangle formed in the larger triangle, so their corresponding angles are equal.

Since the original triangles are similar, the remaining sides must be parallel. This leads to the construction of a convex quadrilateral with angle sum equal to $360°$, which would contradict the previous corollary. ■

3.2. **Parallel Lines in Hyperbolic Geometry:**

In hyperbolic geometry, the concept of parallelism is different from that of Euclidean geometry.

There are two types of parallels in hyperbolic geometry:

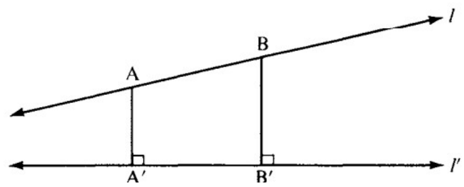

Figure 2. Limitings Parallel lines.

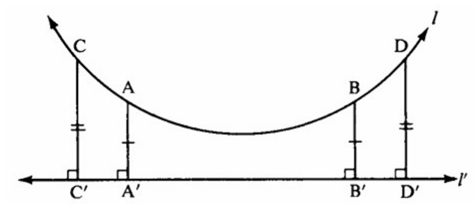

Figure 3. Ultra-pallel lines.

- **Limiting parallels**, which approach the given line as closely as possible without crossing it.
  - A limiting parallel shares a common perpendicular asymptotically with the given line.
  - Two limiting parallels to the same line through the same point form the boundary between intersecting lines and ultra-parallels.
  - Limiting parallels do not meet the given line, but they "approach" it in a precise sense that will become clearer in the models.
- **Ultra-parallels**, which also do not cross the given line, but remain more widely separated from it.
  - Any two ultra-parallel lines have a unique common perpendicular.
  - Ultra-parallels are "truly disjoint," in the sense that they do not get arbitrarily close to each other.

These two kinds of parallels will be easier to visualize once we introduce specific models of hyperbolic geometry. In those models, it will become clear how these lines behave and how they relate to each other.

For now, it is enough to keep in mind that parallelism in hyperbolic geometry is not unique, and this is one of its most distinctive features.

4. Models of Hyperbolic Geometry

To prove the consistency of hyperbolic geometry, we must represent it in terms of Euclidean geometry. This means constructing a model

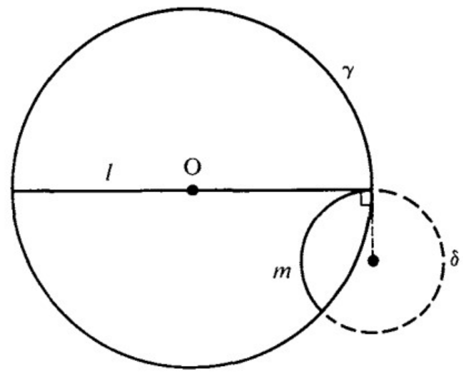

Figure 4. Hyperbolic lines in Disk Model.

where points and lines of the hyperbolic plane correspond to certain Euclidean objects, but where incidence and distance are redefined. If the hyperbolic axioms are satisfied within this model, then any contradiction in hyperbolic geometry would imply a contradiction in Euclidean geometry. This approach does not prove absolute consistency, but it shows that hyperbolic geometry is at least as consistent as the Euclidean system. The models of hyperbolic geometry were developed precisely for this purpose: to provide a concrete interpretation of the hyperbolic world inside the familiar Euclidean space.

4.1. **Poincaré Disk Model:**

The Poincaré disk model represents the hyperbolic plane as the set

$$\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}.$$

Definition 4.1. The hyperbolic lines in the Poincaré disk model are the diameters of the unit circle and the arcs of circles contained in the disk whose centers lie outside the disk and that intersect the boundary of the disk orthogonally.

Definition 4.2. A circle intersects the boundary of the disk orthogonally if, at each intersection point, the tangent to the circle is perpendicular to the tangent to the boundary.

In this model, the ideal points correspond to the points of the boundary circle. The angles between curves inside the disk are equal to the corresponding hyperbolic angles.

4.2. **Poincaré Upper Half-Plane Model:**

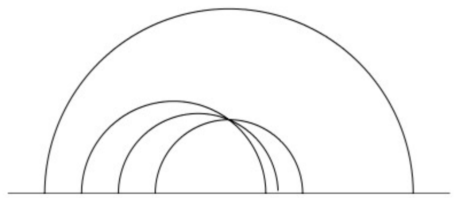

Figure 5. The fifth postulate does not hold.

The Poincaré upper half-plane model represents the hyperbolic plane as the set

$$\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}.$$

Definition 4.3. The hyperbolic lines in this model are the semicircles centered on the real axis that intersect the real axis orthogonally and the vertical lines.

The boundary of the model is the real axis together with the point at infinity; this boundary represents the ideal points. The angles between curves in the upper half-plane are equal to the corresponding hyperbolic angles.

4.3. **Klein Model:**

The Klein model represents the hyperbolic plane as the interior of the disk $\mathbb{D}$

Definition 4.4. The hyperbolic lines in the Klein model are the Euclidean line segments that join two points in the disk and, when extended, intersect the boundary of the disk at two distinct points.

In this model, the ideal points correspond to the points of the boundary circle. The angles measured in the Klein model do not coincide with the hyperbolic angles. However, incidence relations are represented exactly as in Euclidean geometry.

5. An introduction to Hyperbolic Area

Modern hyperbolic geometry is studied through models that help us understand how points, lines, and figures behave in a plane that does not follow the rules of Euclidean geometry. Unlike the classical approach, this perspective also involves transformations that preserve certain features of the space, and introduces tools that allow us to compare and measure figures with precision.

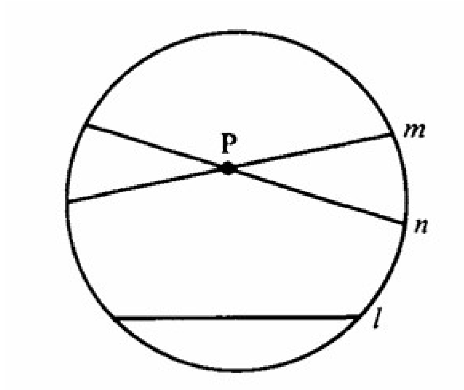

Figure 6. Parallel lines in Klein model.

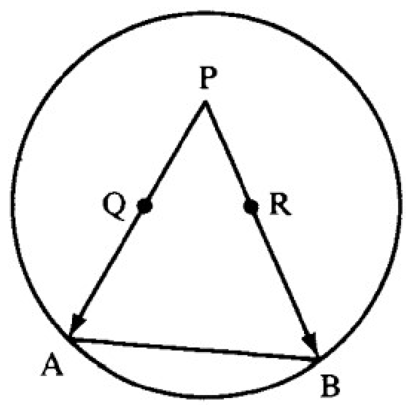

Figure 7. Easy visualization of limiting parallels.

In this section, we step away from the geometric style used so far. We will no longer consider multiple models or construct visual arguments. Instead, we focus on a single model of the hyperbolic plane, the upper half-plane model, because it gives us a clear setting for what comes next. Our goal is to introduce a way of assigning a value to regions in this model and use it to find a general formula for the area of figures such as triangles.

While it might seem natural in other contexts to begin with concepts like distance, curve length, or angle measurement, here we choose to focus directly on area. This is because hyperbolic area has simple and elegant properties that can be established without relying on more technical ideas. Moreover, it is sufficient for the results we aim to present, and allows us to proceed in a direct and effective way. This section is meant to prepare the ground for what follows, and can also serve as an invitation to those interested in exploring more modern aspects of hyperbolic geometry later on.

For a modern and rigorous treatment of hyperbolic geometry, including models, distance, and area, the reader is referred to [And05], [Bea83], and [Rat06].

5.1. Upper Half Plane.

We can also represent the hyperbolic plane using the upper half of the complex plane. For this purpose, consider the set

$$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}.$$

In this model, the hyperbolic lines are either vertical Euclidean lines, which contain the point at infinity, or semicircles orthogonal to the real axis. The boundary of $\mathbb{H}$ is the extended real line $\mathbb{R} \cup \{\infty\}$, and it represents the ideal points of the model.

5.2. Möbius transformations:

Möbius transformations appear naturally in hyperbolic geometry and play a central role in many situations. In this work, we will use them as a tool for working within the upper half-plane model.

Definition 5.1. A Möbius transformation is any function of the form $f(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. This function is defined on $\mathbb{R} \cup \{\infty\}$, with the convention

$$f(\infty) = \begin{cases} \frac{a}{c} & \text{if } c \neq 0 \\ \infty & \text{if } c = 0 \end{cases} \quad \text{and} \quad f(-d/c) = \infty \text{ if } c \neq 0.$$

We say that f *preserves the model* if it maps the upper half-plane $\mathbb{H}$ onto itself.

Example. Some Möbius transformations that preserve $\mathbb{H}$ are:

- $z \mapsto z + b$, horizontal translation with $b \in \mathbb{R}$,
- $z \mapsto kz$, dilation with $k > 0$,
- $z \mapsto -1/z$, inversion.

Proposition 5.2. *Every Möbius transformation that preserves the upper half-plane maps hyperbolic lines to hyperbolic lines.*

Proposition 5.3. *Every Möbius transformation that preserves the upper half-plane preserves angles at points of intersection.*

Proposition 5.4. *Every Möbius transformation that preserves the upper half-plane preserves hyperbolic area.*

5.3. **Area formula in the upper half-plane model.**

To measure area in the upper half-plane model, we use an expression that assigns a positive real value to certain regions in the plane. This assignment is compatible with the transformations that preserve the model: regions that can be transformed into one another by Möbius transformations have the same area.

The formula we will use is the following: if R is a suitable region in the upper half-plane $\mathbb{H}$, its hyperbolic area is given by $\text{Area}(R) = \iint_R \frac{1}{y^2}\,dx\,dy$ where $x = \text{Re}(z)$ and $y = \text{Im}(z)$. This defines a notion of area different from the usual one, and it is adapted to hyperbolic geometry.

We will not explain the origin of this formula. For our purposes, it is enough to know that this expression is compatible with the structure of the model and behaves well under the transformations that preserve $\mathbb{H}$. In particular, every Möbius transformation that maps $\mathbb{H}$ onto itself leaves this area invariant.

This formula will be used later to compute the area of hyperbolic triangles and to support key results in the next sections.

Proposition 5.5. *Let f be a Möbius transformation that maps $\mathbb{H}$ onto itself. Then, for every suitable region $R \subset \mathbb{H}$, Area$(f(R))$ = Area(R), where the area is computed using the formula Area$(R) = \iint_R \frac{1}{y^2}\,dx\,dy$.*

5.4. **Area of hyperbolic triangles in $\mathbb{H}$.**

In this model, a hyperbolic triangle is a region bounded by three different lines that intersect in pairs at distinct points in the upper half-plane or on its boundary. If the triangle is entirely contained in $\mathbb{H}$, it is called a finite triangle; if one or more of its vertices lie on the boundary, it is called an ideal triangle.

Theorem 5.6. *Let T be a hyperbolic triangle in $\mathbb{H}$ with interior angles α, β and γ, measured in radians. Then the hyperbolic area of T is*

$$\text{Area}(T) = \pi - (\alpha + \beta + \gamma).$$

Proof.

Lemma 5.7. *Let T be a hyperbolic triangle in $\mathbb{H}$ with interior angles α, β and γ, measured in radians. Then the hyperbolic area of T is Area$(T) = \pi - (\alpha + \beta + \gamma)$ when one vertex is an ideal vertex.*

Proof. Let A, B, and C be the vertices of a hyperbolic triangle, and suppose that C is an ideal vertex.

If $C = \infty$, then A and B cannot lie on the same vertical line; otherwise, all three points would be collinear along a hyperbolic line, which contradicts the assumption that they form a triangle. Therefore, A and B must lie on a semicircle of radius $r > 0$ centered at some real number c.

Consider the Möbius transformation

$$f(z) = \frac{z - c}{r},$$

which maps the center c of the semicircle to 0 and rescales the radius to 1. The image triangle now has two vertices lying on the semicircle of radius 1 centered at the origin, and the third vertex remains at ∞.

If instead C is a real number $c \in \mathbb{R}$, we apply the Möbius transformation

$$f(z) = \frac{1}{z - c},$$

which maps C to ∞, and sends A and B to two other points in $\mathbb{H}$ that do not lie on the same vertical line (since Möbius transformations preserve non-collinearity of distinct points on hyperbolic lines). Then, we proceed as in the previous case by applying a further transformation of the form

$$z \mapsto \frac{z - c'}{r'}$$

to send the image of A and B onto a semicircle of radius 1 centered at the origin.

Thus, after composing these transformations, we may assume without loss of generality that our triangle has one vertex at ∞, and the other two vertices lying on the semicircle of radius 1 centered at 0, since we are only considering Möbius transformations that preserve the upper half-plane, which in particular preserve both hyperbolic area and angles.

For convenience, we continue referring to the images of A and B after applying these transformations as A and B.

We now continue using the diagram above. The area of our original triangle is equal to the area of the region Δ depicted in the figure. Note that the vertex at ∞ corresponds to a right angle in the Euclidean sense between the vertical lines and the semicircle, which reflects the fact that the angle at that ideal vertex is zero in the hyperbolic geometry. The other two internal angles of the triangle are α and β, and they also appear at the corresponding points where the triangle meets the real axis. These repetitions arise from the fact that Möbius transformations preserve angles, so the transformed triangle retains the same angle values at the corresponding points.

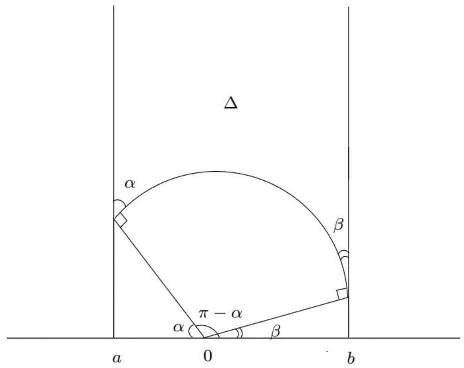

Figure 8. There is at least one ideal vertex.

We now compute the area of the triangle using the standard area element in the upper half-plane model:

$$\mathrm{Area}_{\mathbb{H}}(\Delta) = \iint_{\Delta} \frac{1}{y^2}\, dx\, dy.$$

To evaluate this, we integrate vertically from the semicircle up to infinity, and then horizontally from $x = a$ to $x = b$. This gives:

$$\begin{aligned}\mathrm{Area}_{\mathbb{H}}(\Delta) &= \int_a^b \left(\int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\, dy \right) dx \\ &= \int_a^b -\frac{1}{y}\bigg|_{y=\sqrt{1-x^2}}^{y=\infty} dx \\ &= \int_a^b \frac{1}{\sqrt{1-x^2}}\, dx.\end{aligned}$$

Consider the geometric setup after applying the Möbius transformations. The triangle has one vertex at infinity, and the other two vertices on the semicircle of radius 1 centered at 0. The side AB corresponds to the semicircular arc from A to B, and the side BC corresponds to the vertical line at $x = \cos\beta$ that extends to infinity.

In the upper half-plane model, the hyperbolic angle at B is measured as the Euclidean angle between the tangents to these two curves at B. Since the semicircle has center at the origin, the tangent to the semicircle at B is perpendicular to the radius connecting the origin to B.

The point B has coordinates $(\cos\beta, \sin\beta)$, so the angle between the radius $\overline{OB}$ and the positive real axis is β. Because the tangent is perpendicular to the radius, the angle between the tangent at B and the vertical line $x = \cos\beta$ is exactly β.

Therefore, the hyperbolic angle at B is equal to β. Similarly, the hyperbolic angle at A is equal to α, since $A = (-\cos\alpha, \sin\alpha)$.

Thus, the integral becomes:

$$\int_a^b \frac{1}{\sqrt{1-x^2}}\,dx = \int_{-\cos\alpha}^{\cos\beta} \frac{1}{\sqrt{1-x^2}}\,dx.$$

This is a standard integral that yields:

$$\arcsin(\cos\beta) - \arcsin(-\cos\alpha) = \arcsin(\cos\beta) + \arcsin(\cos\alpha).$$

Since $\arcsin(\cos\theta) = \frac{\pi}{2} - \theta$ for $\theta \in (0, \frac{\pi}{2})$, we obtain:

$$\text{Area}_{\mathbb{H}}(\Delta) = \left(\frac{\pi}{2} - \beta\right) + \left(\frac{\pi}{2} - \alpha\right) = \pi - (\alpha + \beta).$$

This completes the proof in the case when one vertex of the triangle lies on the boundary of $\mathbb{H}$.

■

We now prove the general formula for the area of a hyperbolic triangle with no ideal vertices. Let A, B, and C be the vertices of the triangle, with angles α, β, and γ respectively.

By applying a Möbius transformation, we may assume that A and C lie on the same vertical line, with C directly above A, and that B lies to the right of this line.

Consider the two auxiliary triangles:

- The triangle $AB\infty$, which has angles α at A and $\beta + \delta$ at B, where δ is the angle between BC and the vertical line from B to ∞.
- The triangle $BC\infty$, which has angles $\pi - \gamma$ at C and δ at B.

Since both $AB\infty$ and $BC\infty$ have one ideal vertex at ∞, their areas are given by the standard formula:

$\text{Area}(AB\infty) = \pi - \alpha - (\beta + \delta)$, $\quad \text{Area}(BC\infty) = \pi - (\pi - \gamma) - \delta = \gamma - \delta$.

The triangle ABC is exactly the difference between $AB\infty$ and $BC\infty$:

$\text{Area}(ABC) = \text{Area}(AB\infty) - \text{Area}(BC\infty)$.

Substituting the expressions, we obtain:

$\text{Area}(ABC) = (\pi - \alpha - \beta - \delta) - (\gamma - \delta) = \pi - \alpha - \beta - \gamma$. ■

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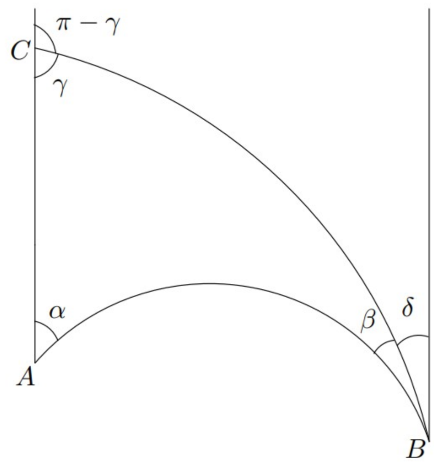

Figure 9. There is no ideal vertex.

Euler Circle, Mountain View, CA 94040
Email address: r.yaranga.almeida@gmail.com