Algebraic Closure of Function Fields

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Introduction

My topic is actually really simple so in all likelihood, probably won't be a long talk.
That being said, if you have actual questions related to the content of the talk, or if I am going to fast, please feel free to let me know. DO NOT be afraid questions, I like teaching and want all of you to understand and appreciate the topic of this paper.

Groups and then...?

Rings are groups with an additional operator that repeats the original operator of the group (addition, multiplication). Not necessarily, but it needs to follow the distributive property. Some times this new operator has a identity element, (a ring with unity) and sometimes this new operation is commutative (a commutative ring).
They're not quite right however. They still form a group under the first operation, but not under the second.
Some groups are rings by default, and all rings are groups by definition.
R[x], or R adjoin x, is the ring of all polynomials with coeffecients in x.

Rings and then...?

Fields are more versatile and less general than rings They have really interesting properties that span way further than this talk. Unfortunately, we won't be talking about any of that.
A field (denoted F) is a commutative ring of unity that forms a group under multiplication, as in, $\forall a \in F, \exists\ a^{-1} \mid a \times a^{-1} = e$ where e is the multiplicative identity element 1.
We can actually turn rings into fields with the **Frac**() operator. **Frac**(R) essentially takes every element in R and divides by ever other element in R setting equivalence relations for fractions that equal one another, and ignoring fractions that are divisions by zero.

$$\textbf{Frac}(\mathbb{Z}) = \mathbb{Q}$$

A function field is a field that has an indeterminate adjoined to it, like $\mathbb{Q}(x)$

Algebraic closure

oftentimes we want to complete fields and rings is by closing them algebraically with a finite or infinite (almost always infinite) amount of splitting fields, for every $P(x) \in F[x]$. The algebraic closure of some field F is denoted $\overline{F}$ as:

$$\overline{F} = \bigcup_{P(x) \in F[x]}^{\infty} \mathbf{SF}_F(P(x))$$

We say that a field K is algebraically closed if all polynomials with coefficients in K also have zeros in K

$$\overline{\mathbb{Q}} = \mathbb{C}$$

$$\overline{\mathbb{Q}} \neq \mathbb{R}$$

because $x^2 + 2$ doesn't have roots in $\mathbb{R}$

Formal Power Series

The formal power series of x with coefficients in R is denoted $R[[x]]$. It is all functions in the form:

$$\sum_{n=0}^{\infty} a_n x^n \mid a_n \in R, n \geq 0$$

Some things to note about this object:

1. $R[x] \subset R[[x]]$
2. Most of the series in this ring **are not convergent**, some are, but most aren't. If you want strictly only the convergent power series, we have a different ring and notation (check the paper).
3. The formal power series **doesn't allow for fractional exponents**, that would be formal puiseux series denoted $R\langle\langle x\rangle\rangle$ or $R[[x^{1/n}]] \mid n \in \mathbb{N}$

Puiseux series

So these are a little weird, but there basically any power series, complex or real, that has fractional exponents of x.

$$\sqrt{2}(x-1)^{1/2} + \frac{\sqrt{2}}{4}(x-1)^{3/2} - \frac{\sqrt{2}}{32}(x-1)^{5/2} + \frac{\sqrt{2}}{128}(x-1)^{7/2} - \dots$$

Formal Power Series

Multiplication in the Formal Power Series:

$$\left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{k=0}^{n+m} c_k x^k$$

where

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}$$

and addition:

$$A = \sum_{n=0}^{N} a_n x^n, B = \sum_{n=0}^{M} b_n x^n$$

$$A + B = \sum_{n=0}^{\max(N,M)} (a_n + b_n) x^n$$

Algebraic closure of function fields, and the Newton-Puiseux theorem

The Newton-Puiseux theorem essentially states that:
For any $f(x,y) \in \mathbb{C}[[x]][[y]]$, the zeros, or liner factors of $f(x,y)$ exist within $\mathbb{C}\langle\langle x\rangle\rangle$ or $\overline{\mathbb{C}[[x]]}$.

$$\overline{\textbf{Frac}(\mathbb{C}[[x]])} = \textbf{Frac}(\mathbb{C}\langle\langle x\rangle\rangle)$$

or more formally,

$$\overline{\textbf{Frac}(\mathbb{C}\langle\langle x\rangle\rangle)} = \textbf{Frac}(\mathbb{C}\langle\langle x\rangle\rangle)$$

Abstract Example

Let us look at a possible $P(x, y)$, and we will organize our terms in a typical binomial fashion:

$$P(x, y) = c_0 y^a + c_1 y^b x^A + c_2 y^c x^B + \cdots + c_n x^n.$$

Let us treat x like a constant. Now we can consolidate our $P(x, y)$ into a one variable function $P(y)$:

$$P(y) = c_x y^a + c_{x1} y^b + c_{x2} y^c + \cdots + c_n.$$

The fundamental Theorem of algebra says that we can split this $P(y)$ into its liner factors, which might resemble:

$$0 = P(y) = (y - a_x)(y - a_{x2})(y - a_{x3}) \cdots (y - a_{xn}).$$

But remember, because each we treated our variable x like a constant and consolidated it, each of our a_{xn} roots are themselves functions of x.

THE END

What about the proof?
What about the algebraic closure of finite fields like

$$\overline{\textbf{Frac}(\mathbb{Z}_p[[x]])}$$

where p is some prime?
What about something unrelated like completion of x-adic metric spaces with $R[x]$?
Go read the paper.
Questions? Commnets? Concerns?