Geometric Group Theory

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Overview

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- Specifically, it is used to explore the connections between the algebraic and geometric properties of groups and spaces.
- Geometric group theory can be used as a means to learn about the properties of geometric objects through groups, or to represent groups themselves geometrically.

Groups

Definition (Group)

A *group* $(G, \star)$ is a set G with a binary operation $\star : G \times G \to G$, satisfying the following axioms:

- *Associativity* – For $g_1, g_2, g_3 \in G$, we have that
$$g_1 \star (g_2 \star g_3) = (g_1 \star g_2) \star g_3.$$
- *Identity* – There exists an identity element e such that for all $g \in G$,
$$g \star e = e \star g = g.$$
- *Inverses* – For each $g \in G$, there exists an inverse element $g^{-1} \in G$ such that
$$g \star g^{-1} = g^{-1} \star g = e.$$

Group Presentations

Definition (Generating set)

A group G is *generated* by a subset $S \subset G$ if every element $g \in G$ can be expressed as a combination under the group operation of finitely many generators $s \in S \cup S^{-1}$.

Definition (Relations)

A *relation* of a group G is a relationship (equality) between some of the generators of G. A *relator* is a term without an equals sign, which is taken to be equal to the identity e.

Group Presentations

Notation (Group Presentation)

We use $\langle S \mid R \rangle$ to denote the group with a generating set S and a set of relations R.

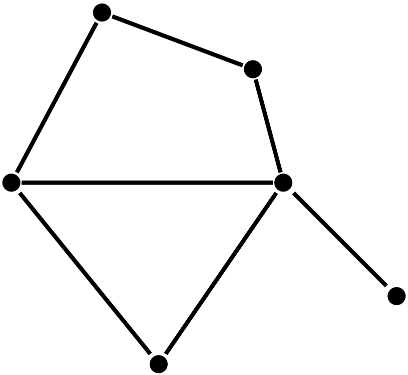
Definition (Finitely generated group)

A *finitely generated group* is a group G with a finite generating set S.

Graphs

Definition (Graph)

A *graph* is a pair of sets (V, E), where the elements of V are *vertices* and the elements of E are *edges*. An edge $e \in E$ is an unordered pair $\{u, v\}$ of vertices $u, v \in V$. In a *directed graph*, the edges are ordered pairs of vertices (u, v).

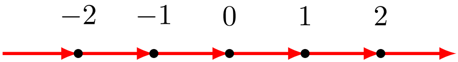


Cayley Graphs

Definition (Cayley graph)

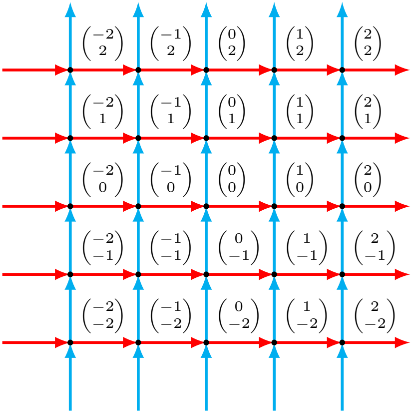
The *Cayley graph* $\Gamma(G, S)$ is a graph of a group $(G, \cdot)$ and its generating set S. Its vertex set is G and its edges are directed $(g, g \cdot s)$ for $g \in G$ and some $s \in S$. The edges of the Cayley graph can also be labeled corresponding to the generator s being used.

Examples of Cayley Graphs



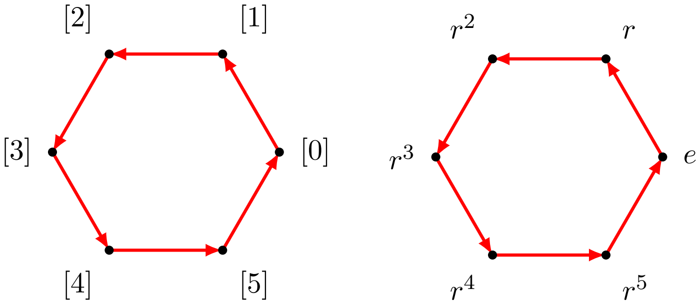
Cayley graph of $\Gamma((\mathbb{Z}, +), \{1\})$.

Examples of Cayley Graphs



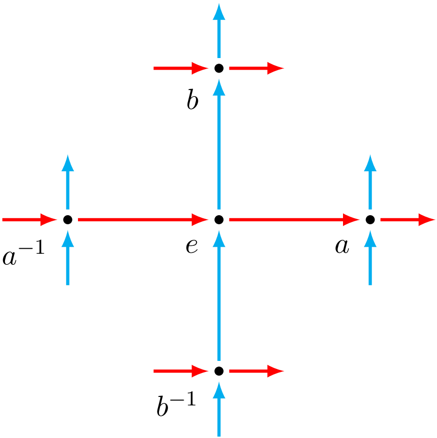
Cayley graph of $\Gamma((\mathbb{Z}^2, +), \{(1,0), (0,1)\})$.

Examples of Cayley Graphs



Cayley graph of $\Gamma((\mathbb{Z}/6\mathbb{Z},+),\{[1]\}) \cong \Gamma(C_6,\{r\})$.

Examples of Cayley Graphs



Cayley graph of $\Gamma(F_2 = \langle\{a, b\} \mid \varnothing\rangle, \{a, b\})$.

Word Metric

Definition (Word)

A *word* w is a concatenated sequence $s_1 s_2 \ldots s_L$ of a group G's generators S and their inverses S^{-1}.
The *length* of the word w is the integer L.
The *evaluation* of the word w is the element $\bar{w}$ given by applying the group operation to the entries of the word in order.

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Definition (Word metric)

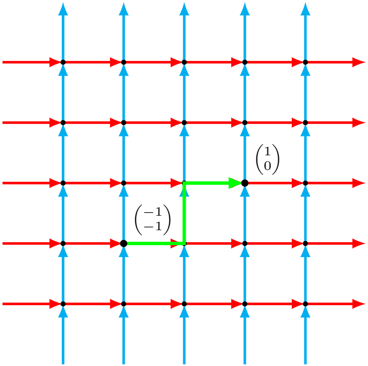
The *word metric* d_S on a group $(G, \cdot)$ with generating set $S \subset G$ is associated with the Cayley graph $\Gamma(G, S)$. The distance between elements $g, h \in G$ is given by

$$d_S(g,h) = \min\left\{L \in \mathbb{N} : g^{-1}h = s_1 \cdots s_L, s_i \in S \cup S^{-1}\right\}.$$

Connection to Cayley Graphs

- The word metric provides a way to make measurements within a group, while the Cayley graph provides a visual geometric representation.
- The distance $d_S(g,h)$ is actually equivalent to the shortest path between the elements g and h on the Cayley graph.

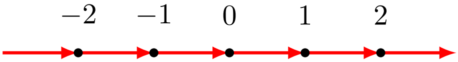
Connection to Cayley Graphs



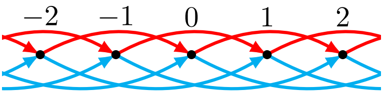
Example

The distance $d_S((-1,-1),(1,0)) = 3$ in the above Cayley graph. In fact, the Cayley graph of $\mathbb{Z}^2$ is the taxicab geometry, and the word metric is the Manhattan distance.

Example of Quasi-Isometric Groups



Cayley graph of $\Gamma((\mathbb{Z},+),\{1\})$.



Cayley graph of $\Gamma((\mathbb{Z},+),\{2,3\})$.

Definition of Quasi-Isometry

Definition (Quasi-isometry)

Let $f : X \to Y$ be a map between metric spaces (X, d_X) and (Y, d_Y). The map f is a *quasi-isometry* if the following properties hold:

- The map f is a (a, b)-*quasi-isometric embedding* if there are constants $a \geq 1$, $b \geq 0$ such that for all $x, x' \in X$,

$$\frac{1}{a} \cdot d_X(x, x') - b \leq d_Y\left(f(x), f(x')\right) \leq a \cdot d_X(x, x') + b.$$

- Every point of Y is within a constant distance $c \geq 0$ of an image point, so for all $y \in Y$, there exists $x \in X$ such that

$$d_Y(y, f(x)) \leq c.$$

Quasi-Geodesics

Definition (Geodesic space)

Let (X, d) be a metric space. A *geodesic* of length $L \geq 0$ is an isometric embedding $\gamma : [0, L] \to X$. The points $\gamma(0)$ and $\gamma(L)$ are the start and end points of the geodesic, respectively.
The space is a *geodesic space* if for all $x, x' \in X$, there exists a geodesic in X with start point x and end point x'.

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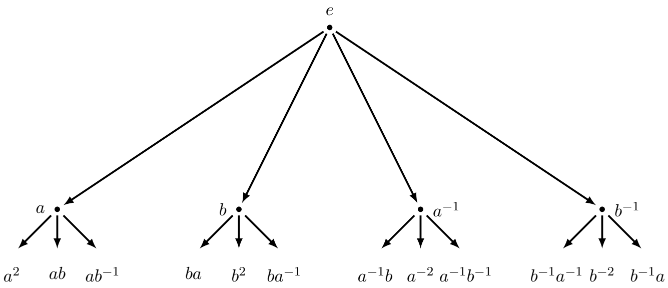
Definition (Quasi-geodesic space)

A (a, b)-*quasi-geodesic* in X is a (a, b)-quasi-isometric embedding $\gamma : [t, t'] \to X$, where $\gamma(t)$ is the start point of γ and $\gamma(t')$ is the end point.
The space is a *quasi-geodesic space* if for all $x, x' \in X$, there exists a (a, b)-quasi-geodesic in X with start point x and end point x'.

Example of Group Growth



Growth of F_2.

Growth Functions

Definition (Growth function)

Let G be a finitely generated group with a generating set $S \subset G$. The ball of radius r around the identity is

$$B_{G,S}(r) = \{g \in G \mid d_S(g, e) \leq r\}.$$

The *growth function* of G with respect to S is $\beta_{G,S} : \mathbb{N} \to \mathbb{N}$ where

$$\beta_{G,S}(r) = |B_{G,S}(r)|.$$

Types of Growth

Definition (Growth types)

A group has a

- *Polynomial growth rate* if

$$\beta_{G,S}(r) \leq Cr^k.$$

- *Exponential growth rate* if

$$\beta_{G,S}(r) \geq a^r.$$

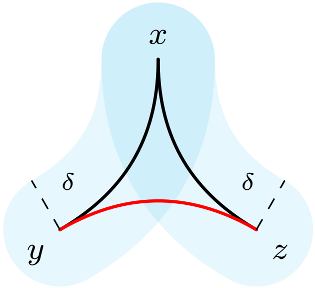
- *Intermediate growth rate* if

$$Cr^k \leq \beta_{G,S}(r) \leq a^r.$$

Hyperbolic Spaces

Definition (δ-slim triangle)

Let X be a metric space and $x, y, z \in X$. A *geodesic triangle* is the union of the geodesic segments $[x, y]$, $[x, z]$, and $[y, z]$. If for any $p \in [y, z]$, there is a point in $[x, y] \cup [x, z]$ a distance less than δ away, then the triangle is *δ-slim*.



Definition (Hyperbolic space)

A space X is *δ-hyperbolic* if all geodesic triangles in X are δ-slim. A space is *hyperbolic* if there exists a $\delta \geq 0$ such that X is δ-hyperbolic.

Hyperbolic Groups

Definition (Hyperbolic group)

Let G be a finitely generated group with generating set $S \subset G$. Let X be the corresponding Cayley graph, which is a metric space with the word metric. The group G is a *hyperbolic group* if X is a hyperbolic space.

The Word Problem

Definition (Word problem)

Let $\langle S \mid R \rangle$ be a presentation for a group G. The *word problem* for the presentation is solvable if there is an algorithm for every input word w with entries in $S \cup S^{-1}$ that can decide whether w represents the identity element e.

Based on work by Mikhail Gromov and Max Dehn, it is known that hyperbolic groups have solvable word problems.

Summary

Geometric group theory is a very useful for learning about the algebraic and geometric properties of groups and spaces.

- *Cayley graphs* provide a geometric representation of a group, in terms of the graph and its word metric.
- *Quasi-isometries* examine the large-scale geometric properties of groups.
- *Growth* of groups analyze the Cayley graph and comparison between groups.
- *Hyperbolic groups* are a type of group related to hyperbolic geometry.

These are only some of the many aspects of geometric group theory, which have applications to other fields as well, within and outside of mathematics.

Thank You!