GEOMETRIC GROUP THEORY

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ABSTRACT. This paper gives an introduction to the subject of geometric group theory. Geometric group theory is a field which relates the algebraic properties of groups with the geometric properties of spaces. Key concepts include a group's Cayley graph and its word metric, group actions on a set, quasi-isometries between groups, the growth of groups, and hyperbolic groups. The paper provides an overview of each topic with important definitions and examples showing how each relies on the others to build many of their main ideas and are relevant to geometry and group theory in many interesting and applicable ways.

1. Introduction

1.1. **Overview.** Geometric group theory is an area of abstract algebra that studies finitely generated groups. Specifically, it is used to explore the connections between the algebraic and geometric properties of groups and spaces. Geometric group theory can be used as a means to learn about the properties of geometric objects through groups, or to represent groups themselves geometrically.

1.2. **Historical Background.** Geometric group theory is a relatively new branch of mathematics. It began to develop more towards the end of the 20th century and has roots in the field of combinatorial group theory, which studied discrete groups with group presentations. Related topics were first studied around the mid and late 19th century, by mathematicians such as William Rowan Hamilton and Walther von Dyck. In the first half of the 20th century, the work of those such as Max Dehn, Jakob Nielson, Otto Schreier, and Egbert van Kampen helped introduce some geometric and topological ideas into the field. Geometric group theory really emerged as a new and distinct area after Mikhail Gromov's 1987 monograph

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"Hyperbolic Groups", which was influential to the field as it set up new ideas that led to the use of the term "geometric group theory".

1.3. **Outline.** This paper provides an overview of some of the fundamental concepts and key ideas in geometric group theory, with various definitions and examples within those topics as well as explanations of how they are related to each other notions in geometry and group theory.

- This paper begins with a few introductory definitions and preliminary information related to group theory, as well as examples of some types of groups relevant to geometric group theory.
- Then, Cayley graphs of groups are discussed, which are a way to geometrically visualize a group, as well as a group's word metric, which offers a way to measure distances in a group with respect to the generating set.
- Next, we discuss group actions, which
- Following that, we explore quasi-isometries, another notion of distance and the similarity between groups in terms of large-scale geometry.
- After that, we examine group growth
- Finally, we go over hyperbolic groups, which are a specific type of group whose Cayley graph and word metric satisfy properties related to hyperbolic geometry.

2. Preliminaries

Before delving into geometric group theory, we will begin with some preliminary definitions and background with respect to group theory in general.

2.1. **Groups.**

Definition 2.1 (Group)**.** A *group* $(G, \star)$ is a set G with a binary operation $\star : G \times G \to G$, satisfying the following axioms:

- *Associativity* – For $g_1, g_2, g_3 \in G$, we have that

$$g_1 \star (g_2 \star g_3) = (g_1 \star g_2) \star g_3.$$

- *Identity* – There exists an identity element e such that for all $g \in G$,

$$g \star e = e \star g = g.$$

- *Inverses* – For each $g \in G$, there exists an inverse element $g^{-1} \in G$ such that

$$g \star g^{-1} = g^{-1} \star g = e.$$

Definition 2.2 (Generating set)**.** A group G is *generated* by a subset $S \subset G$ if every element $g \in G$ can be expressed as a combination of finitely many *generators*, that is $s_1 s_2 \cdots s_n$ for $s_i \in S \cup S^{-1}$.

Definition 2.3 (Relations)**.** A *relation* of a group G is a relationship (equality) between some of the generators of G. A *relator* is a term without an equals sign, which is taken to be equal to the identity e.

Notation (Group Presentation)**.** We use $\langle S \mid R \rangle$ to denote the group with a generating set S and a set of relations R.

Definition 2.4 (Finitely generated group)**.** A *finitely generated group* is a group G with a finite generating set S.

2.2. **Types of Groups.** Here are some examples of a few different types of groups. These groups are relatively common in group theory and have connections to geometry as well. They will also be used in various examples throughout this paper, as they are relevant to the upcoming topics.

2.2.1. *Symmetry Groups.*

Definition 2.5 (Symmetry group)**.** A *symmetry group* is the group of all geometric transformations on an object which leave it invariant. The group operation is composition of these transformations.

Definition 2.6 (Cyclic group)**.** A *cyclic group* is a group that can be generated by a single element. It is denoted by C_n and has the presentation $\langle g \mid g^n = e \rangle$. A cyclic group can also represent the n-fold rotational symmetries of an object.

Definition 2.7 (Dihedral group)**.** A *dihedral group* is the group of symmetries of a regular polygon, which consists of rotations and reflections.

Example. Figure 1 shows the eight symmetries of a square. These symmetries form the dihedral group $D_4 = \langle r, s \mid r^4, s^2, rs = sr^{-1} \rangle$, a symmetry group with order 8.

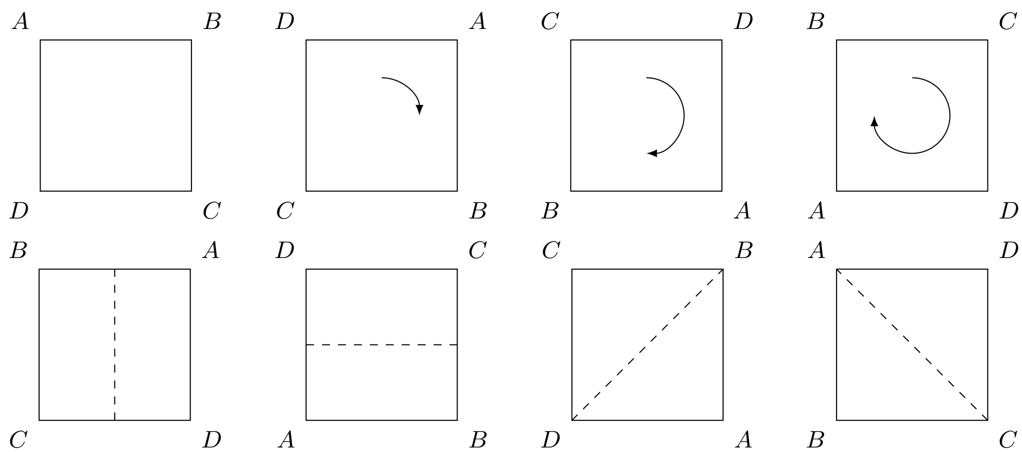

Figure 1. Symmetries of a square.

2.2.2. *Permutation Groups.*

Definition 2.8 (Permutation)**.** A *permutation* is a bijection $\sigma : X \to X$ from a set X to itself. A permutation is essentially just a reordering of the elements of a set into a new arrangement.

Notation (Word representation)**.** A permutation can be written as a *word representation*, which is just the resultant sequence after the permutation is applied to the set's natural ordering.

Notation (Cycle notation)**.** A permutation σ can be represented in *cycle notation.* A cycle is written as $(x\ \sigma(x)\ \sigma(\sigma(x))\ \cdots\ \sigma^{-1}(x))$, and multiple cycles are written together until all elements have been written. 1-cycles such as (x) can also be omitted in the notation.

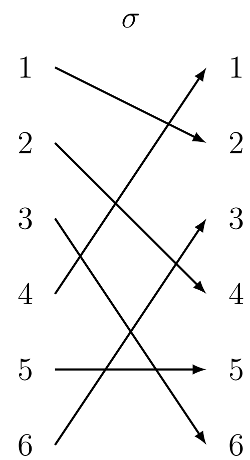

Figure 2. A permutation mapping σ.

Example. Figure 2 shows a permutation of the set $\{1, 2, 3, 4, 5, 6\}$. The word representation would be $\sigma = 246153$. In cycle notation, $\sigma = (124)(36)$.

Definition 2.9 (Permutation group)**.** A *permutation group* is a group G whose elements are permutations of a given set X. The group operation is composition of these permutations.

Definition 2.10 (Symmetric group)**.** The *symmetric group* $\mathrm{Sym}(X)$ is the group of all permutations of X. If $X = \{1, 2, \dots, n\}$, then $\mathrm{Sym}(X)$ can be denoted S_n, which has $n!$ elements.

2.2.3. *Free Groups.*

Definition 2.11 (Word)**.** A *word* w is a concatenated sequence $s_1 s_2 \dots s_n$ of a group G's generators S and their inverses S^{-1}.

The *length* of the word w is the integer n.

The *reduction* of the word w involves removing redundant pairs of generators and their inverses, such as gg^{-1} or $g^{-1}g$, which simplifies the word but keeps the corresponding group element the same.

The *evaluation* of the word w is the element $\bar{w}$ given by applying the group operation to the entries of the word in order.

Definition 2.12 (Free group)**.** A *free group* F_S over a generating set S is the group of all words that can be made from S. The group operation is concatenation of the generators, followed by reduction if necessary. The *rank* of a free group is the number of generators, and F_n can be used to denote a free group of rank n. Free groups are "free" because there are no relations between elements, apart from trivial relations from group axioms.

Example. For example, the free group F_2 has the presentation $\langle\{a, b\} \mid \varnothing\rangle$.

2.2.4. *Integer Groups.*

Definition 2.13 ($\mathbb{Z}$)**.** The group of integers $(\mathbb{Z}, +)$ is the set $\mathbb{Z}$ of all integers and the additive operator $+$.

Addition is usually the operator used for integer groups, so the groups can be written with the operator and ordered pair, for example as $\mathbb{Z}$.

Definition 2.14 ($\mathbb{Z}/n\mathbb{Z}$)**.** The group $\mathbb{Z}/n\mathbb{Z}$ is integers modulo n under the operation of addition modulo n. The elements are the equivalence classes of the integers modulo n, and are denoted as $\{[0], [1], \ldots, [n-1]\}$.

The group $\mathbb{Z}/n\mathbb{Z}$ is a cyclic group of order n, while the group $\mathbb{Z}$ is an infinite cyclic group. Therefore $\mathbb{Z}$ is also a free group of rank 1.

Definition 2.15 ($\mathbb{Z} \times \mathbb{Z}$)**.** The *cartesian product* $\mathbb{Z} \times \mathbb{Z}$ of sets of integers forms a group under addition. The product $\mathbb{Z} \times \mathbb{Z}$ can be denoted as $\mathbb{Z}^2$, and in general $\mathbb{Z}^n$ for higher dimensions.

Definition 2.16 ($\mathbb{Z} * \mathbb{Z}$)**.** The *free product* $\mathbb{Z} * \mathbb{Z}$ of sets of integers forms a group under addition. The free product essentially combines the two groups such that the elements are concatenated and reduced words from both groups.

The free product $\mathbb{Z} * \mathbb{Z}$ is also isomorphic to the free group F_2.

3. Cayley Graphs

3.1. **Cayley Graphs.** Cayley graphs are a powerful tool for visually representing groups. Like graphs in general, Cayley graphs can be embedded in a space (e.g. a 2D plane), which makes studying them geometrically much easier.

Definition 3.1 (Graph)**.** A *graph* is a pair of sets (V, E), where the elements of V are *vertices* and the elements of E are *edges*. An edge $e \in E$ is an unordered pair $\{u, v\}$ of vertices $u, v \in V$.

Definition 3.2 (Directed graph)**.** A *directed graph* is similar to a graph, except the edges E are ordered pairs (u, v) for $u, v \in V$. This introduces a sense of "direction" to the edges, which are also called *arcs* in a directed graph.

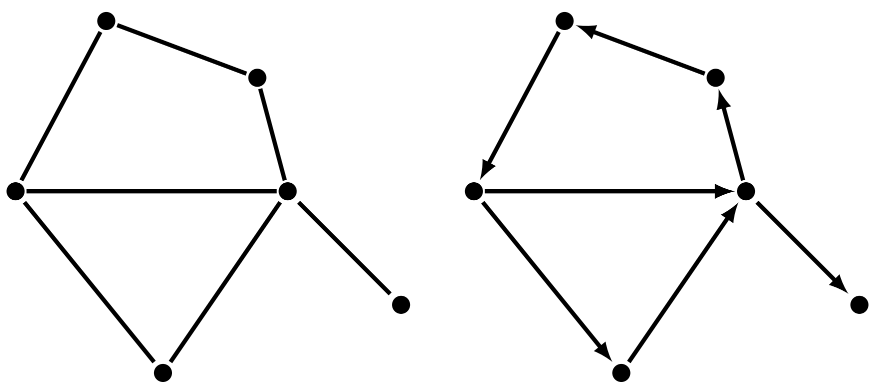

Figure 3. A graph and a directed graph.

Definition 3.3 (Cayley graph)**.** The *Cayley graph* $\Gamma(G, S)$ is a graph of a group $(G, \cdot)$ and its generating set S. Its vertex set is G and its edges are directed $(g, g \cdot s)$ for $g \in G$ and some $s \in S$. The edges of the Cayley graph can also be labeled corresponding to the generator s being used.

See Appendix A for diagrams of Cayley graphs for different groups.

3.2. **Word Metrics.** While the Cayley graph provides a visual geometric representation, the word metric provides a way to make measurements within a group.

Definition 3.4 (Metric space)**.** A *metric space* is a pair (X, d) of a set X and a *metric* d which is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties for all $x, y, z \in X$:

- $d(x, y) = 0$ if and only if $x = y$, and otherwise $d(x, y) > 0$.
- The distance $d(x, y) = d(y, x)$ is symmetric.
- The triangle inequality holds, so $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 3.5 (Word metric)**.** The *word metric* d_S on a group $(G, \cdot)$ with generating set $S \subset G$ is associated with the Cayley graph $\Gamma(G, S)$. The distance between elements $g, h \in G$ is given by

$$d_S(g, h) = \min \left\{ n \in \mathbb{N} : g^{-1}h = s_1 \cdots s_n, s_i \in S \cup S^{-1} \right\}.$$

A useful fact which relates the Cayley graph and the word metric is that the distance $d_S(g, h)$ is actually equivalent to the shortest path between the elements g and h on the Cayley graph.

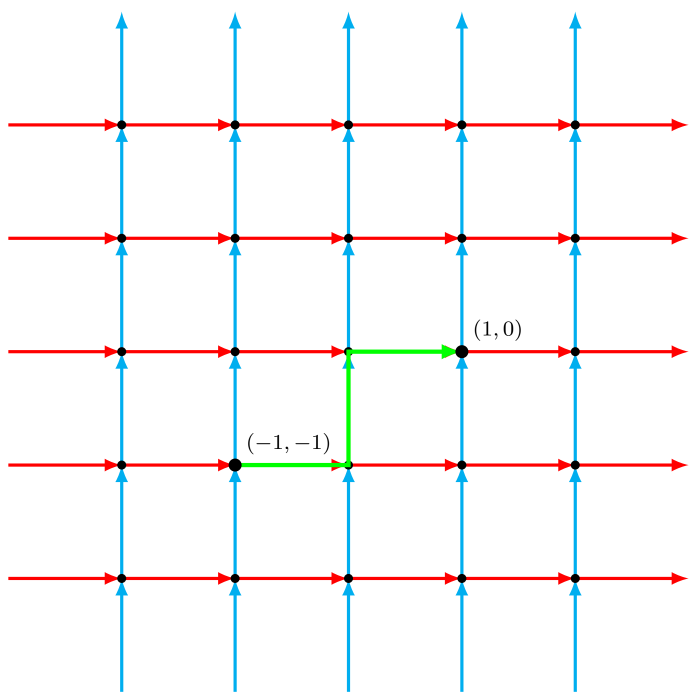

Figure 4. Path between $(-1, -1)$ and $(1, 0)$ on Cayley graph of $\Gamma(\mathbb{Z}^2, \{(1, 0), (0, 1)\})$.

Example. The distance $d_S((-1,-1),(1,0)) = 3$ in the Cayley graph in Figure 4. In fact, the Cayley graph of $\mathbb{Z}^2$ is what is known as the taxicab geometry, and the word metric is the related Manhattan distance.

4. Group Actions

Group actions are a way to view a group G as the group of symmetries of a set X, that is, the symmetric group $\mathrm{Sym}(X)$. The elements of G can act on the set X in a way that preserves certain properties, similar to geometric symmetries

4.1. **Group Actions.**

Definition 4.1 (Group Action)**.** A *group action* of a group G on a set X is a map $\cdot : G \times X \to X$ satisfying the following properties:

- *Identity* – For all $x \in X$, $e \cdot x = x$.
- *Compatability* – For all $g, h \in G$ and all $x \in X$, $(g \cdot h) \cdot x = g \cdot (h \cdot x)$.

Definition 4.2 (Free action)**.** A group action is *free* if for all $x \in X$ and all $g \in G$, $g \cdot x = x$ only if $g = e$.

Definition 4.3 (Transitive actions)**.** A group action is *transitive* if for any two elements $x, y \in X$, there exists $g \in G$ such that $g \cdot x = y$.

Definition 4.4 (Orbit)**.** Let G be a group acting on a set X. The *orbit* of an element $x \in X$ is

$$G \cdot x = \{g \cdot x \mid g \in G\}.$$

Definition 4.5 (Stabilizer)**.** Let G be a group acting on a set X. The *stabilizer group* of an element $x \in X$ is

$$G_x = \{g \in G \mid g \cdot x = x\}.$$

The orbit describes the set of possible configurations of a point under transformations, while the stabilizers are points that are fixed under the transformation.

4.2. **Actions on Trees.**

Definition 4.6 (Group actions on graphs)**.** A group G acts on a graph (V, E) by the homomorphism $G \to \mathrm{Aut}((V, E))$ with the properties from Definition 4.1 for $g, h \in G$ and $x \in V$ or $x \in E$. Also, if there is an edge $e \in E$ between vertices $v, w \in V$, then $g \cdot e \in E$ is an edge between $g \cdot v, g \cdot w \in V$.

Definition 4.7 (Tree)**.** A *tree* is a connected graph that does not contain any cycles. A *connected graph* is a graph with paths between every pair of vertices, and a *cycle* is a non-empty path which starts and ends at the same vertex.

Theorem 4.8. *A group is free if and only if it acts freely on a tree.*

Although we won't cover the proof of this theorem, a direct consequence of it is Theorem 4.9.

Corollary 4.9 (Nielson-Schreier Theorem)**.** *Every subgroup of a free group is free.*

Proof. For a free group G and a subgroup $H \subset G$, the group G acts freely on a tree and thus so does the subgroup H. Therefore, by Theorem 4.8, H is also a free group. ■

Theorem 4.10 (Orbit-Stabilizer Theorem)**.** *For a finite group G acting on a set X with $x \in X$,*

$$|G \cdot x| = [G : G_x].$$

5. Quasi-Isometries

Quasi-isometries are another way to look at the metric properties of a group, with focus especially on large-scale geometric structure.

5.1. **Quasi-isometries.**

Definition 5.1 (Quasi-isometry)**.** Let $f : X \to Y$ be a map between metric spaces (X, d_X) and (Y, d_Y).

The map f is a *quasi-isometry* if the following properties hold:

- The map f is a (a,b)-*quasi-isometric embedding* if there are constants $a \geq 1$, $b \geq 0$ such that for all $x, x' \in X$,

$$\frac{1}{a} \cdot d_X(x,x') - b \leq d_Y\left(f(x), f(x')\right) \leq a \cdot d_X(x,x') + b.$$

- Every point of Y is within a constant distance $c \geq 0$ of an image point, so for all $y \in Y$, there exists $x \in X$ such that

$$d_Y(y, f(x)) \leq c.$$

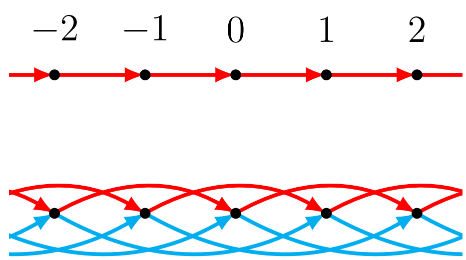

Figure 5. Cayley graph of $\Gamma(\mathbb{Z}, \{1\})$ and $\Gamma(\mathbb{Z}, \{2, 3\})$.

Example. For example, in Figure 5, the Cayley graphs of the two groups would look similar if zoomed out, since they are both quasi-isometric.

Example. The groups $\mathbb{Z}$ and $\mathbb{R}$ as well as the groups $\mathbb{Z}^2$ and $\mathbb{R}^2$ are quasi-isometric.

5.2. **Quasi-geodesics.**

Definition 5.2 (Geodesic space)**.** Let (X, d) be a metric space. A *geodesic* of length $L \geq 0$ is an isometric embedding $\gamma : [0, L] \to X$. The points $\gamma(0)$ and $\gamma(L)$ are the start and end points of the geodesic, respectively.

The space is a *geodesic space* if for all $x, x' \in X$, there exists a geodesic in X with start point x and end point x'.

Definition 5.3 (Quasi-geodesic space)**.** A (a,b)-*quasi-geodesic* in X is a (a,b)-quasi-isometric embedding $\gamma : [t, t'] \to X$, where $\gamma(t)$ is the start point of γ and $\gamma(t')$ is the end point.

The space is a *quasi-geodesic space* if for all $x, x' \in X$, there exists a (a, b)-quasi-geodesic in X with start point x and end point x'.

6. Group Growth

Somewhat related to the idea of metrics on groups and their Cayley graphs is the study of group growth. The growth of a group is based the size or "volume" of a ball centered around the identity, and the growth of the ball as the radius increases provides insight into the group's structure.

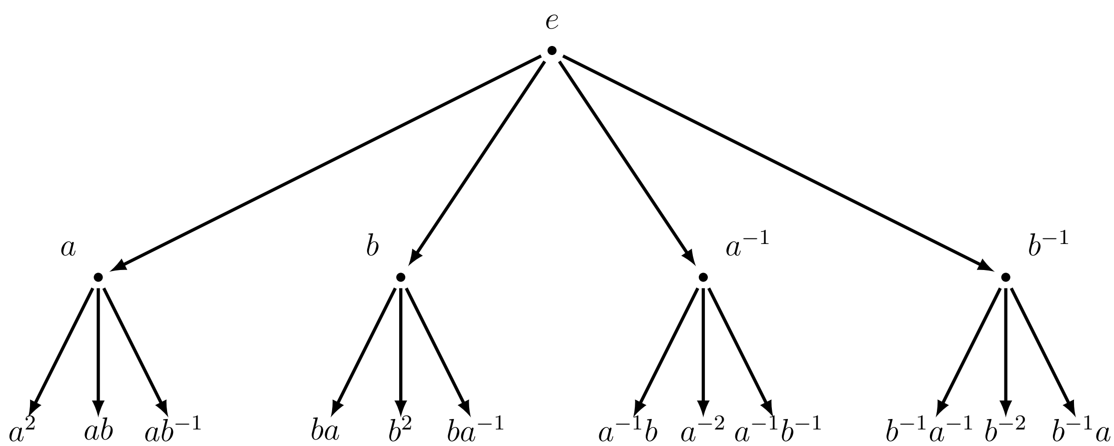

Figure 6. Growth of F_2.

6.1. Growth Functions.

Definition 6.1 (Growth function)**.** Let G be a finitely generated group with a generating set $S \subset G$. The ball of radius r around the identity is

$$B_{G,S}(r) = \{g \in G \mid d_S(g, e) \leq r\}.$$

The *growth function* of G with respect to S is $\beta_{G,S} : \mathbb{N} \to \mathbb{N}$ where

$$\beta_{G,S}(r) = |B_{G,S}(r)|.$$

6.2. Growth Types.

Definition 6.2 (Growth types)**.** A group has a

- *Polynomial growth rate* if
$$\beta_{G,S}(r) \leq Cr^k.$$
- *Exponential growth rate* if
$$\beta_{G,S}(r) \geq a^r.$$
- *Intermediate growth rate* if
$$Cr^k \leq \beta_{G,S}(r) \leq a^r.$$

Example. For example, the growth of F_2 as shown in Figure 6 is exponential.

7. Hyperbolic Groups

Hyperbolic geometry is a type of non-Euclidean geometry where every point is a saddle point and curvature is negative, as opposed to being zero like in Euclidean geometry or positive like in elliptic geometry.

7.1. **Hyperbolic Space.**

A hyperbolic space is a metric space satisfying properties related to hyperbolic geometry. Gromov first introduced the definition, along with the use of large-scale hyperbolicity to study hyperbolic groups.

Definition 7.1 (δ-slim triangle)**.** Let X be a metric space and $x, y, z \in X$. A *geodesic triangle* is the union of the geodesic segments $[x, y]$, $[x, z]$, and $[y, z]$.

If for any $p \in [y, z]$, there is a point in $[x, y] \cup [x, z]$ a distance less than $\delta \geq 0$ away, then the triangle is *δ-slim*.

Figure 7 gives an example of a geodesic triangle that is δ-slim, since all points on any of its sides are within a distance of δ from the other two, as shown with the red side $[y, z]$ being contained completely withing the blue region.

Definition 7.2 (Hyperbolic space)**.** A space X is *δ-hyperbolic* if all geodesic triangles in X are δ-slim. A space is *hyperbolic* if there exists a $\delta \geq 0$ such that X is δ-hyperbolic.

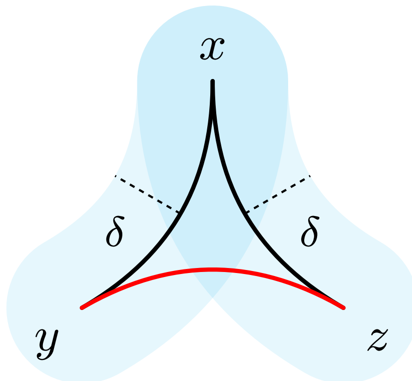

Figure 7. A δ-slim triangle.

In relation to quasi-isometries, the notion of a *quasi-hyperbolic space* can also be defined analogously to a hyperbolic space, but using quasi-geodesics instead of regular geodesics.

7.2. **Hyperbolic Groups.** Now we discuss the hyperbolic group, also developed by Gromov from classical hyperbolic geometry.

Definition 7.3 (Hyperbolic group)**.** Let G be a finitely generated group with generating set $S \subset G$. Let X be the corresponding Cayley graph, which is a metric space with the word metric. The group G is a *hyperbolic group* if X is a hyperbolic space.

Definition 7.4 (Word problem)**.** Let $\langle S \mid R \rangle$ be a presentation for a group G. The *word problem* for the presentation is solvable if there is an algorithm for every input word w with entries in $S \cup S^{-1}$ that can decide whether w represents the identity element e.

Based on work by Gromov and Dehn, it is known that hyperbolic groups have solvable word problems.

8. Conclusion

8.1. **Summary.** Geometric group theory is a very useful for learning about the algebraic and geometric properties of groups and spaces.

- *Cayley graphs* provide a geometric representation of a group, in terms of the graph and its *word metric.*
- *Group actions* allow groups to be seen through symmetries.

- *Quasi-isometries* examine the large-scale geometric properties of groups.
- *Growth* of groups analyze the Cayley graph and comparison between groups.
- *Hyperbolic groups* are a type of group related to hyperbolic geometry.

These are only some of the many aspects of geometric group theory,

8.2. **Applications.** The topics covered in this paper also have applications to other fields as well, within and outside of mathematics. This includes some of the more obvious applications, such as general group theory and geometry, but also areas such as topology and number theory. It is also useful in physics and computer science, for example, with respect to symmetric crystal structures and computational geometry.

Acknowledgements

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Appendix A. Cayley Graph Examples

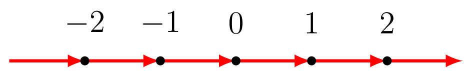

Figure 8. Cayley graph of $\Gamma(\mathbb{Z}, \{1\})$.

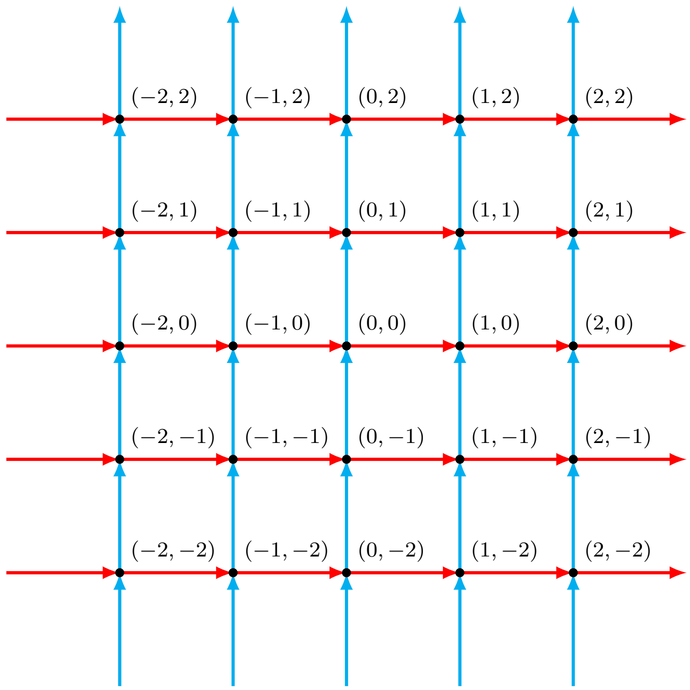

Figure 9. Cayley graph of $\Gamma(\mathbb{Z}^2, \{(1,0), (0,1)\})$.

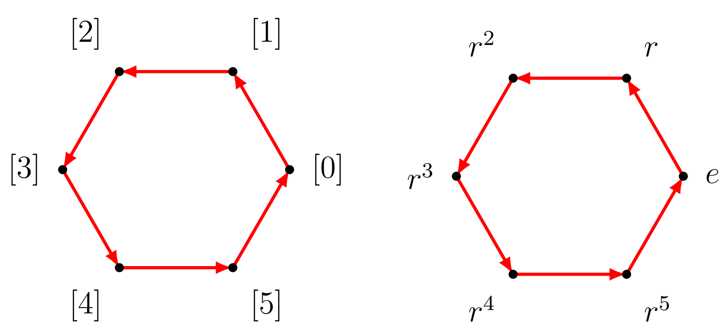

Figure 10. Cayley graph of $\Gamma(\mathbb{Z}/6\mathbb{Z}, \{[1]\}) \cong \Gamma(C_6, \{r\})$.

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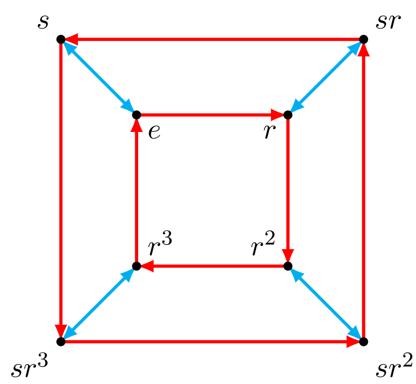

Figure 11. Cayley graph of $\Gamma(D_4, \{r, s\})$.

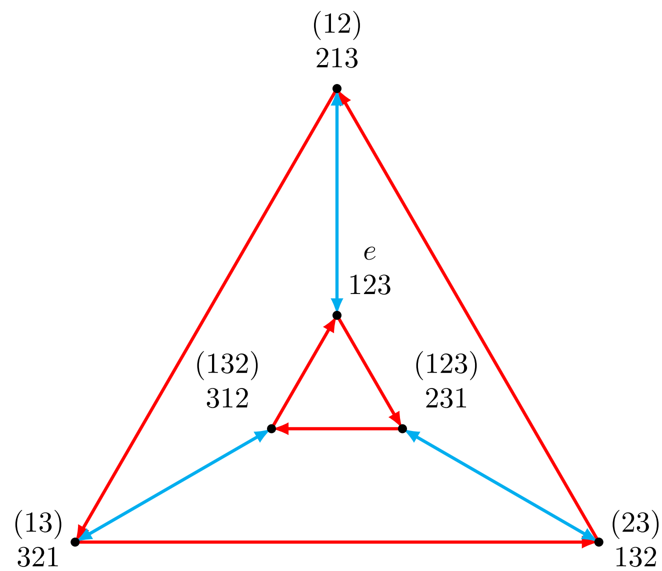

Figure 12. Cayley graph of $\Gamma(S_3 \cong D_3, \{(123), (12)\})$.

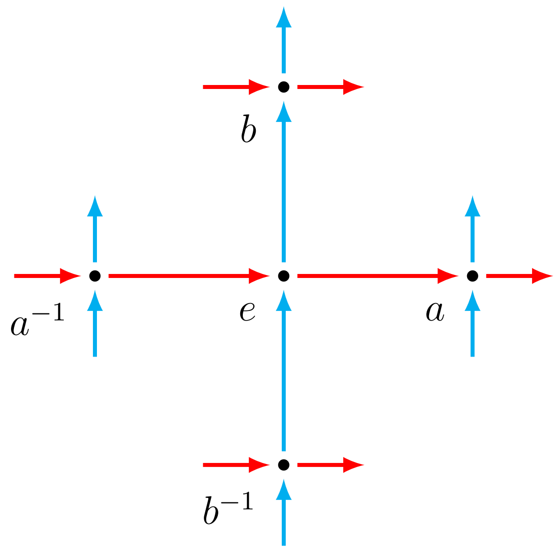

Figure 13. Cayley graph of $\Gamma(F_2, \{a, b\})$.