On the Law of the Iterated Logarithm

Richard Mai

Euler Circle

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Motivation

- Randomness is everywhere: weather, stock markets, NBA tanking seasons, or fumbling your AP test.

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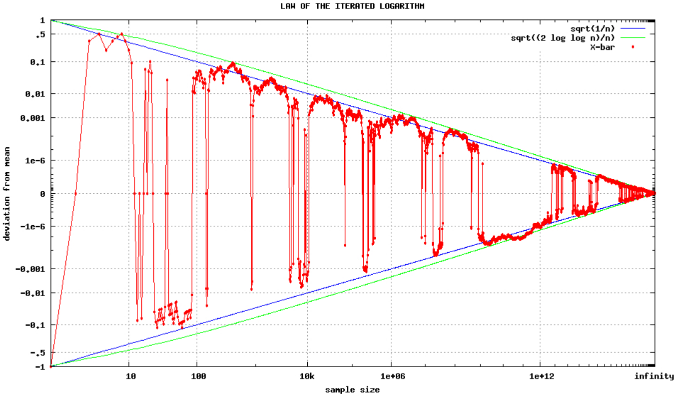
- Random walk seems chaotic, but it has structure.
- LIL tells us the boundary for how far randomness can go.

Law of the Iterated Logarithm

Formally, let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent, identically distributed random variables with $\mathbb{E}[X_i] = 0$ and $\mathrm{Var}(X_i) = 1$, and let $S_n = X_1 + X_2 + \cdots + X_n$. Then the Law of the Iterated Logarithm states:

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1 \quad \text{almost surely.}$$

The LIL Visualization



- LIL describes the maximal oscillations of partial sums S_n.
- Unlike LLN or CLT, it characterizes extreme events.

Law of Large Numbers

The Strong Law of Large Numbers(SLLN) states that the average of the independent and identically distributed (i.i.d.) Random Variables converges almost surely to the expected value, that is,

$$Pr(\lim_{n\to\infty} \frac{X_1 + X_2 + X_3 + X_4 \ldots X_n}{n} = \mu) = 1,$$

where μ is the expected value.

Central Limit Theorem

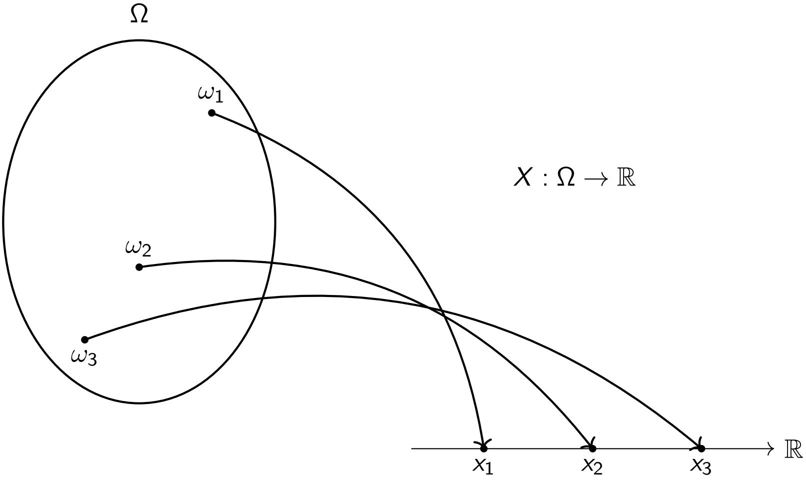
The Central limit theorem, or the CLT, states that for large n, the distribution of the random variable $\bar{X}_n$ after standardization, meaning centered at zero, and divide by the standard deviation of $\bar{X}_n$, converges to the normal distribution $\mathcal{N}(0, 1)$.
In other words, as $n \to \infty$,

$$\sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right) \to \mathcal{N}(0, 1).$$

Central Limit Theorem

Random Variables

A random variable is a function that maps random events to a number; it is a way of quantifying events to study them.



Inequality Toolbox

- **Markov's Inequality**:

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

- **Chebyshev's Inequality**:

$$\mathbb{P}(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$$

- **Chernoff's Bound**:

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[e^{tX}]}{e^{ta}}$$

The Borel-Cantelli Lemmas

First Borel-Cantelli Lemma:

- If $\sum \mathbb{P}(E_n) < \infty$, then $\mathbb{P}(E_n \text{ occurs i.o.}) = 0$
- Rare events with summable probabilities occur only finitely often.

Second Borel-Cantelli Lemma:

- If $\sum \mathbb{P}(E_n) = \infty$ and the E_n are independent, then $\mathbb{P}(E_n \text{ occurs i.o.}) = 1$
- Frequent events with diverging probabilities occur infinitely often.

Maximal Inequality

Let $S_n = X_1 + X_2 + \cdots + X_n$, where $\{X_j\}_{j=1}^{\infty}$ are independent and identically distributed (i.i.d.) random variables with zero mean ($E[X_j] = 0$) and unit variance ($E[X_j^2] = 1$). (And implicitly all moments of X_j exist and can be determined by taking the derivative of the moment generating function). Suppose that the positive constant sequence $a_1, a_2, \ldots, a_n, \ldots$ satisfies $a_n \to \infty$ and $\frac{a_n}{\sqrt{n}} \to 0$.

Then there exists a sequence $\zeta_1, \zeta_2, \ldots$ with $\zeta_n \to 0$ such that

$$P[S_n \geq a_n\sqrt{n}] = e^{-a_n^2(1+\zeta_n)/2}.$$

Maximal Inequality

Let $S_n = X_1 + \cdots + X_n$, where $\{X_j\}_{j=1}^{\infty}$ are independent and identically distributed (i.i.d.) random variables with mean 0 and variance 1. (And all moments of X_j exist, which can be determined by taking the derivatives of its moment generating function). Also, let $\Pr[S_0 = 0] = 1$.
Then for $\alpha \geq \sqrt{2}$,

$$\Pr\left[\max\{S_0, S_1, \ldots, S_n\} \geq \alpha\sqrt{n}\right] \leq 2\Pr\left[\frac{S_n}{\sqrt{n}} \geq \alpha - \sqrt{2}\right].$$

Proof Strategy Sketch

- Prove upper bound: $\limsup \leq 1$ using **First Borel-Cantelli Lemma**
- Prove lower bound: $\limsup \geq 1$ using **Second Borel-Cantelli Lemma**
- Key idea: show how often deviations above/below threshold happen

Definition of lim sup

$$\Pr\left[\limsup_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{\sigma\sqrt{2n \log\log n}} = 1\right] = 1$$

it suffices to prove that for every positive ϵ, (where ϵ is understood as "countable" in its range),

$$P\left(\left\{x \in \mathbb{R}^\infty : (\exists N)(\forall n > N) \frac{x_1 + x_2 + \cdots + x_n}{\sigma\sqrt{2n \log\log n}} - 1 < \epsilon\right\}\right) = 1$$

and

$$P\left(\left\{x \in \mathbb{R}^\infty : (\forall N)(\exists n > N) \frac{x_1 + x_2 + \cdots + x_n}{\sigma\sqrt{2n \log\log n}} - 1 > -\epsilon\right\}\right) = 1$$

Equivalently,

$$P\left(\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\left\{x\in\mathbb{R}^{\infty}:\frac{x_1+x_2+\cdots+x_n}{\sigma\sqrt{2n\log\log n}}-1\geq\epsilon\right\}\right)=0$$

and

$$P\left(\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\left\{x\in\mathbb{R}^{\infty}:\frac{x_1+x_2+\cdots+x_n}{\sigma\sqrt{2n\log\log n}}-1>-\epsilon\right\}\right)=1.$$

Upper Bound Outline

- Let $\theta > 1$ be a fixed constant. Define a subsequence $n_k = \lfloor \theta^k \rfloor$ to sparsely sample the indices.
- This spacing ensures that the growth of n_k is exponential, which helps in bounding the probability of rare events.
- Using large deviation estimates:

$$\mathbb{P}(S_{n_k} \geq (1+\epsilon)\sqrt{2n_k \log\log n_k}) \approx e^{-c \log k} = \frac{1}{k^c}$$

- Since $\sum \mathbb{P}(S_{n_k} \geq (1+\epsilon)\sqrt{2n_k \log\log n_k}) < \infty$, by the first Borel-Cantelli Lemma,
- $\Rightarrow$ With probability 1, only finitely many S_{n_k} exceed $(1+\epsilon)\sqrt{2n_k \log\log n_k}$.

Lower Bound Outline

- Choose subsequence differences $S_{n_k} - S_{n_{k-1}}$
- Construct a_k such that:

$$\sum \mathbb{P}(S_{n_k} - S_{n_{k-1}} \geq a_k \sqrt{m_k}) = \infty$$

- Apply Borel-Cantelli #2 $\Rightarrow$ event happens infinitely often

Why It Matters

- Reveals structure in extreme randomness
- Has applications in: finance, ML, queueing, network theory
- A peak achievement in 20th century probability

Thank you!