Diophantine Tuples and Baker's Theorem

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July 15, 2025

What is a Diophantine Tuple?

- A *Diophantine* $D(1)$*-tuple* is a set of positive integers $\{a_1, a_2, \ldots, a_k\}$ such that for all $i \neq j$,

$$a_i a_j + 1 \quad \text{is a perfect square.}$$

- Example: $\{1, 3, 8, 120\}$ is a $D(1)$-quadruple.
- Natural question: can we extend $\{1, 3, 8, 120\}$ to a quintuple?

Our Goal

- Show that if $\{1, 3, 8, d\}$ is a $D(1)$-quadruple, then $d = 120$ is the only possibility.
- Approach:
  - Translate to equations involving squares.
  - Use Pell-type equations to parametrize solutions.
  - Apply transcendence theory (Baker's theorem) to bound possible solutions.
  - Final finite check confirms only $d = 120$.

System of Equations for $\{1, 3, 8, d\}$

We require integers x, y, z, d such that:

$$d + 1 = x^2, \quad 3d + 1 = y^2, \quad 8d + 1 = z^2.$$

From the first two equations:

$$3x^2 - y^2 = 2.$$

From the first and third:

$$z^2 - 8x^2 = -7.$$

We want to find all integers x, y, z satisfying these simultaneously.

Definition: Pell Equation and Norm

Definition (Pell Equation)

A *Pell equation* is a Diophantine equation

$$x^2 - D\,y^2 = N,$$

where $D > 0$ is a fixed non-square integer and $N \in \mathbb{Z}$.

Definition (Norm)

For $\alpha = x + y\sqrt{D} \in \mathbb{Q}(\sqrt{D})$, the *norm* is

$$\mathrm{Norm}(\alpha) = x^2 - D\,y^2.$$

Fundamental Solution and Generating All Solutions

Definition (Fundamental Solution)

The *fundamental solution* of

$$x^2 - D\,y^2 = 1$$

is the least positive integer pair (x_1, y_1). The element

$$\varepsilon = x_1 + y_1\sqrt{D}$$

is called the *fundamental unit*.

Definition (Generating Solutions)

If $\alpha_0 = x_0 + y_0\sqrt{D}$ satisfies $\mathrm{Norm}(\alpha_0) = N$, then all integer solutions to

$$x^2 - D\,y^2 = N$$

are given by

$$x_n + y_n\sqrt{D} = \alpha_0\,\varepsilon^n, \quad n \in \mathbb{Z}.$$

Pell Equation 1: $3x^2 - y^2 = 2$

- Rewrite as $y^2 - 3x^2 = -2$.
- Fundamental solution: $(y_1, x_1) = (2, 1)$.
- Fundamental unit in $\mathbb{Q}(\sqrt{3})$:

$$\varepsilon = 2 + \sqrt{3}.$$

- All solutions given by:

$$\alpha_n = y_n + x_n\sqrt{3} = (1 + \sqrt{3})\varepsilon^n.$$

$$x_n = \frac{\alpha_n - \overline{\alpha_n}}{2\sqrt{3}} = \frac{(1+\sqrt{3})\varepsilon^n - (1-\sqrt{3})\varepsilon^{-n}}{2\sqrt{3}}$$

$$y_n = \frac{\alpha_n + \overline{\alpha_n}}{2} = \frac{(1+\sqrt{3})\varepsilon^n + (1-\sqrt{3})\varepsilon^{-n}}{2}.$$

Pell Equation 2: $z^2 - 8x^2 = -7$

- Fundamental solution: $(z_1, x_1) = (3, 1)$.
- Fundamental unit in $\mathbb{Q}(\sqrt{8})$:

$$\eta = 3 + \sqrt{8}.$$

- All solutions given by:

$$\alpha_m = z_m + x_m\sqrt{8} = (1 + \sqrt{8})\eta^m.$$

$$x_m = \frac{\beta_m - \overline{\beta_m}}{2\sqrt{8}} = \frac{(1+\sqrt{8})\eta^m - (1-\sqrt{8})\eta^{-m}}{2\sqrt{8}}$$

$$z_m = \frac{\beta_m + \overline{\beta_m}}{2} = \frac{(1+\sqrt{8})\eta^m + (1-\sqrt{8})\eta^{-m}}{2}$$

Matching the Two Parametrizations

To have a common x, we require:

$$x_n = x_m,$$

where

$$x_n = \frac{A\,\varepsilon^n - A'\,\varepsilon^{-n}}{C}, \qquad x_m = \frac{B\,\eta^m - B'\,\eta^{-m}}{D},$$

with

$$A = 1 + \sqrt{3}, \quad A' = 1 - \sqrt{3}, \quad C = 2\sqrt{3},$$
$$B = 1 + \sqrt{8}, \quad B' = 1 - \sqrt{8}, \quad D = 2\sqrt{8}.$$

Multiplying both sides by CD gives

$$D\bigl(A\,\varepsilon^n - A'\,\varepsilon^{-n}\bigr) = C\bigl(B\,\eta^m - B'\,\eta^{-m}\bigr).$$

Matching Dominant Terms in $x_n = x_m$

We begin with:

$$x_n = x_m \quad \Longrightarrow \quad \frac{A\,\varepsilon^n - A'\,\varepsilon^{-n}}{C} = \frac{B\,\eta^m - B'\,\eta^{-m}}{D},$$

where

$$\begin{aligned} \varepsilon &= 2+\sqrt{3}, \quad A = 1+\sqrt{3}, \quad A' = 1-\sqrt{3}, \quad C = 2\sqrt{3}, \\ \eta &= 3+\sqrt{8}, \quad B = 1+\sqrt{8}, \quad B' = 1-\sqrt{8}, \quad D = 2\sqrt{8}. \end{aligned}$$

Multiply both sides by CD and isolate dominant terms:

$$D\,A\,\varepsilon^n = C\,B\,\eta^m \;+\; \underbrace{D\,A'\,\varepsilon^{-n} \;-\; C\,B'\,\eta^{-m}}_{\text{small error}}.$$

Divide by $CB\,\eta^m$:

$$\frac{DA}{CB}\,\frac{\varepsilon^n}{\eta^m} \;=\; 1+\delta, \quad \delta \ll \varepsilon^{-n} - \eta^{-m}.$$

Defining the Key Linear Form Λ

Taking logarithms:

$$\Lambda := n \log \varepsilon - m \log \eta + \log \left(\frac{DA}{CB} \right) = \log(1 + \delta).$$

Since $\delta \ll \varepsilon^{-n} - \eta^{-m}$, and $\log(1 + \delta) \approx \delta$ for small δ, we have:

$$|\Lambda| \approx |\delta| \ll \max(\varepsilon^{-n}, \eta^{-m}).$$

Why define Λ?

- We will apply Baker's theorem to get a lower bound on $|\Lambda|$ that depends only on n, m.
- Combined with the above upper bound, this forces (n, m) into a finite range.

Degree and Absolute Logarithmic Height

Definition (Degree)

Let α be an algebraic number. Its *degree* d is the degree of its minimal polynomial over $\mathbb{Q}$.

Definition (Absolute Logarithmic Height)

Let α be an algebraic number of degree d with conjugates $\alpha_1, \ldots, \alpha_d$ and minimal polynomial $a_d x^d + \cdots + a_0 \in \mathbb{Z}[x]$. Then:

$$h(\alpha) = \frac{1}{d}\left(\log|a_d| + \sum_{i=1}^{d} \log\max\{1, |\alpha_i|\}\right).$$

Baker's Theorem (Matveev's Version for Two Logarithms)

- Let $\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 \neq 0$, where:
  - α_1, α_2 are nonzero algebraic numbers in a number field of degree D.
  - $b_1, b_2 \in \mathbb{Z}$, not both zero.
- Define:

$$A_i \geq \max\{D \cdot h(\alpha_i), |\log \alpha_i|, 0.16\}, \quad B = \max\{|b_1|, |b_2|, 3\}.$$

- Then:

$$|\Lambda| > \exp\left(-C_0 \cdot D^2 \cdot A_1 \cdot A_2 \cdot \log B\right),$$

where $C_0 = 1.13 \times 10^9$ is an explicit constant.

Applying Baker's Theorem to the Linear Form Λ

- Recall the linear form:

$$\Lambda := n \log \varepsilon - m \log \eta + \log \left(\frac{DA}{CB} \right),$$

where

$$\varepsilon = 2 + \sqrt{3}, \quad \eta = 3 + \sqrt{8},$$

and

$$A = 1 + \sqrt{3}, \quad C = 2\sqrt{3},$$
$$B = 1 + \sqrt{8}, \quad D = 2\sqrt{8}.$$

- By Baker's theorem:

$$|\Lambda| > \exp(-C^\star \log \max\{n, m\}).$$

- From the Diophantine equation and the small error term, we also have

$$|\Lambda| \ll \max\left(\varepsilon^{-n}, \eta^{-m}\right).$$

- Combining bounds on $|\Lambda|$ yields an explicit finite bound on $n, m \leq 10^{10}$

Refinement: Baker–Davenport Lemma

- Baker's theorem shows too many (n, m) to check
- Baker–Davenport lemma refines this using continued fractions to reduce the search drastically.
- Let $\theta = \frac{\log \varepsilon}{\log \eta}$, and let $\frac{p_k}{q_k}$ be the k-th convergent in the continued fraction expansion of θ.
- If there is a rational approximation $\frac{p}{q}$ with

  $$\left|\theta - \frac{p}{q}\right| < \frac{1}{2q^2}$$

  and the denominator $q > 6q_k$, then q must be equal to either q_{k+1} or q_{k+2}.
- Thus, to find all good approximations with denominator up to some bound, it suffices to check only convergents and their immediate successors.
- This drastically reduces the candidates for (n, m).

Final Check

- Exhaust all small n, m and check convergents near θ.
- Only solution satisfying both parametrizations:

$$x = 11 \quad \Rightarrow \quad d = x^2 - 1 = 120.$$

- **Therefore, the only extension of $\{1, 3, 8\}$ to a Diophantine quadruple is with $d = 120$.**

Extensions to $D(n)$ and Known Results

Definition ($D(n)$-tuple)

A set of distinct positive integers $\{a_1, a_2, \ldots, a_k\}$ is called a $D(n)$-*tuple* if

$$a_i a_j + n \text{ is a perfect square for all } i \neq j.$$

- $D(1)$-quintuples do not exist (He, Togbé, Ziegler, 2019).
- For general n, the situation is more flexible:
  - (Dujella) If $n \not\equiv 2 \pmod 4$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists at least one $D(n)$-quadruple.
  - For some n, $D(n)$-quintuples and even sextuples are known such as $(99, 315, 9920, 32768, 44460, 19534284)$ is a $D(2985984 = 2^{12}3^6)$-sextuple (Gibbs)
- Diophantine tuples can also be studied over $\mathbb{Q}$