Diophantine Tuples

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Introduction

A *Diophantine m-tuple* is a set of m positive integers $\{a_1, a_2, \ldots, a_m\}$ such that

$$a_i a_j + 1$$

is a perfect square for every $1 \le i < j \le m$. The study of such sets has deep historical roots. Rational examples appeared in the works of Diophantus of Alexandria, but the first known integer quadruple satisfying this property was discovered much later by Fermat:

$$\{1, 3, 8, 120\}.$$

Fermat's set remains central to the theory of Diophantine tuples, as it exhibits a remarkable structure and appears to be maximal. Indeed, no integer can be added to the set $\{1, 3, 8, 120\}$ while preserving the Diophantine condition.

In the 20th century, Baker and Davenport (1969) provided the first proof that this particular quadruple cannot be extended to a quintuple [1]. Their method relied on Diophantine approximation and a careful analysis of solutions to Pell equations. This initiated a long line of research into the existence and classification of Diophantine tuples. Over the following decades, the theory was developed further by many authors, including Dujella, Filipin, Fujita, and others, who applied increasingly refined techniques to rule out the existence of Diophantine quintuples.

The general problem was resolved in full by He, Togbé, and Ziegler in 2016, who proved that no Diophantine quintuple of positive integers exists [6]. Their proof combined deep results from transcendental number theory with extensive computational analysis, culminating in the definitive classification of Diophantine m-tuples over the integers for $m \le 5$.

In this paper, we revisit the classical triple

$$(1, 3, 8)$$

and provide a proof that it extends *uniquely* to the quadruple $(1, 3, 8, 120)$ and cannot be extended to a quintuple. Our approach draws on ideas from both the original Baker–Davenport method and modern techniques. Compared to the original, our argument requires less computational effort and benefits from sharper theoretical bounds, through the use of refined height estimates and explicit transcendence results.

Our proof is based on three main tools:

- An explicit lower bound for a two-term linear form in logarithms (Matveev's refinement of Baker's theorem) (Theorem 2.2),
- Growth estimates for solutions of Pell equations (Lemma 3.1),
- The Baker–Davenport reduction lemma for Diophantine approximation (Lemma 4.3).

Section 1 presents a version of Baker's theorem suitable for our setting, giving explicit lower bounds for linear forms in logarithms. In Section 2, we derive upper bounds on the size of potential solutions by analyzing the growth of Pell equations and fundamental units. Section 3 revisits the Baker–Davenport method, providing an effective reduction argument. Finally, Section 4 applies all of these tools to the specific case of the triple $(1, 3, 8)$ and shows that the only possible extension is 120.

While no Diophantine quintuple exists for $D = 1$, the situation becomes more flexible when one studies $D(n)$-*tuples*, where the condition is that

$$a_i a_j + n$$

is a perfect square for all $1 \leq i < j \leq m$. In this more general setting:

- Dujella showed that if $n \not\equiv 2 \pmod 4$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists at least one $D(n)$-quadruple [3].
- For some values of n, even quintuples and sextuples are known. For instance, the set

  $$(99, 315, 9920, 32768, 44460, 19534284)$$

  forms a $D(2985984)$-sextuple, where $2985984 = 2^{12} \cdot 3^6$, constructed by Gibbs [5].
- Diophantine tuples can also be studied over $\mathbb{Q}$, where infinitely many rational Diophantine sextuples exist [4].

Thus, while the case $D = 1$ is now more resolved, the theory of Diophantine tuples remains an active and richly structured area of number theory.

1 Pell Equations

Definition 1.1 (Pell Equation)**.** A *Pell equation* is a Diophantine equation

$$x^2 - D\,y^2 = N,$$

where $D > 0$ is a fixed nonsquare integer and $N \in \mathbb{Z}$ is fixed.

Definition 1.2 (Norm)**.** If $\alpha = x + y\sqrt{D} \in \mathbb{Q}(\sqrt{D})$, then its *norm* is

$$\mathrm{Norm}(\alpha) = \alpha\,\overline{\alpha} = x^2 - D\,y^2,$$

where $\overline{\alpha} = x - y\sqrt{D}$ is the Galois conjugate.

Definition 1.3 (Fundamental Solution & Unit). The *fundamental solution* of

$$x^2 - D\,y^2 = 1$$

is the least positive integer pair (x_1, y_1). The element

$$\varepsilon = x_1 + y_1\sqrt{D}$$

is the *fundamental unit* of $\mathbb{Q}(\sqrt{D})$.

Proposition 1.4. *Every solution (x_n, y_n) to*

$$x^2 - D\,y^2 = 1$$

satisfies

$$x_n + y_n\sqrt{D} = \varepsilon^n, \quad n \in \mathbb{Z}.$$

More generally, if $\alpha_0 = x_0 + y_0\sqrt{D}$ has $\mathrm{Norm}(\alpha_0) = N$, then all integer solutions of $x^2 - Dy^2 = N$ are

$$x_n + y_n\sqrt{D} = \alpha_0\,\varepsilon^n, \quad n \in \mathbb{Z}.$$

1.1 Reduction to Pell–Type Equations for $(1, 3, 8, d)$

The system

$$d + 1 = x^2, \qquad 3d + 1 = y^2, \qquad 8d + 1 = z^2$$

yields two independent Pell equations:

1. From $d + 1 = x^2$ and $3d + 1 = y^2$:

$$3x^2 - y^2 = 2.$$

   Writing $\alpha = y + x\sqrt{3}$ we have $\mathrm{Norm}(\alpha) = -2$. A minimal solution is $(x_1, y_1) = (1, 1)$, so $\alpha_0 = 1 + \sqrt{3}$ and $\varepsilon = 2 + \sqrt{3}$. Thus

$$y_n + x_n\sqrt{3} = \alpha_0\,\varepsilon^n, \quad x_n = \frac{\varepsilon^n + \varepsilon^{-n}}{2},\ y_n = \frac{\varepsilon^n - \varepsilon^{-n}}{2\sqrt{3}}.$$

2. From $d + 1 = x^2$ and $8d + 1 = z^2$:

$$z^2 - 8x^2 = -7.$$

   A minimal solution is $(x_1, z_1) = (1, 3)$, so $\beta_0 = 1 + \sqrt{8}$ has $\mathrm{Norm}(\beta_0) = -7$, and the fundamental unit is $\eta = 3 + \sqrt{8}$. Thus

$$z_m + x_m\sqrt{8} = \beta_0\,\eta^m, \quad x_m = \frac{(3 + \sqrt{8})^m + (3 - \sqrt{8})^m}{2}.$$

1.2 Matching Parametrizations and Defining Λ

To have a common d we require

$$x_n^2 - 1 = x_m^2 - 1 \quad \Longrightarrow \quad x_n = x_m,$$

i.e.

$$\frac{\varepsilon^n + \varepsilon^{-n}}{2} \;=\; \frac{(3+\sqrt{8})^m + (3-\sqrt{8})^m}{2}.$$

Isolating dominant terms gives

$$\frac{\varepsilon^n}{2}\Big(1 + O(\varepsilon^{-2n})\Big) = \frac{\eta^m}{2}\Big(1 + O(\eta^{-2m})\Big),$$

where $\eta = 3 + \sqrt{8}$. Hence

$$\Lambda := n \ln \varepsilon \;-\; m \ln \eta \;=\; \ln\big(1 + O(\varepsilon^{-2n})\big) \;-\; \ln\big(1 + O(\eta^{-2m})\big)$$

and in particular

$$0 < |\Lambda| \ll \max(\varepsilon^{-2n},\, \eta^{-2m}).$$

This prepares the application of Baker's lower bound on $|\Lambda|$, which will force n, m into a finite range.

2 Proof of Simplified Baker's Theorem

Let K be a number field of degree $D = [K : \mathbb{Q}]$, and fix an embedding

$$\sigma : K \hookrightarrow \mathbb{C}.$$

Let $\alpha_1, \alpha_2 \in K^\times$ be nonzero algebraic numbers such that $\sigma(\alpha_i) \neq 1$, and let $b_1, b_2 \in \mathbb{Z}$, not both zero. We define the linear form in logarithms

$$\Lambda = b_1 \log \sigma(\alpha_1) + b_2 \log \sigma(\alpha_2),$$

where log denotes the principal branch of the complex logarithm.

Definition 2.1 (Absolute Logarithmic Height)**.** Let $\alpha \in \overline{\mathbb{Q}}$ be an algebraic number of degree $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$, with minimal polynomial

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x],$$

and conjugates $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$. Then the absolute logarithmic height of α is defined by

$$h(\alpha) = \frac{1}{d}\left(\log |a_d| + \sum_{i=1}^{d} \log \max\{1, |\alpha_i|\}\right).$$

This height is well-defined, independent of the choice of minimal polynomial, and satisfies basic functoriality properties such as $h(\alpha^m) = |m| h(\alpha)$ for any $m \in \mathbb{Z}$.

Theorem 2.2 (Simplified Baker's Theorem). *Let $\alpha_1, \alpha_2 \in K^\times \setminus \{1\}$ and $b_1, b_2 \in \mathbb{Z}$, not both zero. Let*

$$\Lambda = b_1 \log \sigma(\alpha_1) + b_2 \log \sigma(\alpha_2),$$

as above. Define:

$$A_i \;\geq\; \max\{D\,h(\alpha_i),\; |\log\sigma(\alpha_i)|,\; 0.16\}, \qquad B \;=\; \max\{|b_1|, |b_2|, 3\}.$$

Then there exists a computable absolute constant $C > 0$ such that

$$|\Lambda| \;>\; \exp\left(-C\,D^2\,A_1 A_2 \log B\right).$$

Remark 2.3. This is a two-term case of a general linear form in logarithms. The constant 0.16 ensures that $\log A_i$ remains bounded away from zero in auxiliary estimates. The embedding σ fixes a single archimedean place, and all logarithms are taken in $\mathbb{C}$ using this fixed branch. For the purposes of effective lower bounds, we treat $\Lambda \in \mathbb{C}$, not as a vector in the full logarithmic space.

2.1 Preliminary Lemmas

Lemma 2.4 (Siegel's Lemma [8]). *Let $M \in \mathbb{Q}^{r\times s}$ have rank $r < s$, and suppose each entry has logarithmic height $\leq H$. Then there exists a nonzero $\mathbf{c} \in \mathbb{Z}^s$ with $M\mathbf{c} = 0$ and*

$$\|\mathbf{c}\|_\infty \leq 2\exp\Big(\tfrac{r(rs+1)}{s-r}H + \tfrac{r\log(3s) - s\log(2)}{s-r}\Big).$$

Proof. **Step 1: Clear denominators.**
Write each $m_{ij} = a_{ij}/b_{ij}$ with $\log\max\{|a_{ij}|, |b_{ij}|\} \leq H$, so $|m_{ij}| \leq e^H$. Let

$$D = \operatorname{lcm}(b_{ij}),$$

then

$$\log D \;\leq\; \sum_{i,j} \log|b_{ij}| \;\leq\; rs\,H,$$

and $M' := D\,M \in \mathbb{Z}^{r\times s}$ has $|M'_{ij}| \leq D \cdot e^H = e^{(rs+1)H}$.

Step 2: Count domain vectors.
For $B > 0$, define

$$\mathcal{V} = \{c = (c_1, c_2, \ldots, c_s) \in \mathbb{Z}^s : \|c\|_\infty \leq B\},$$

where the *infinity norm* (or ℓ^∞-norm) of a vector $c \in \mathbb{R}^s$ is defined by

$$\|c\|_\infty := \max_{1\leq i\leq s} |c_i|.$$

In other words, $\mathcal{V}$ consists of all integer vectors whose coordinates are bounded in absolute value by B.

Since each coordinate c_i takes values in the integer interval $[-B, B]$, the cardinality of $\mathcal{V}$ is

$$|\mathcal{V}| = (2B+1)^s > (2B)^s.$$

Step 3: Bound the image.
Each $\mathbf{c} \in \mathcal{V}$ satisfies $\|\mathbf{c}\|_\infty \le B$, so each entry of the product $M'\mathbf{c}$ is a sum of at most s terms of the form $M'_{ij}c_j$. Since $|c_j| \le B$ and $|M'_{ij}| \le e^{(rs+1)H}$ by construction, we have:

$$|(M'\mathbf{c})_i| = \left|\sum_{j=1}^{s} M'_{ij}c_j\right| \le \sum_{j=1}^{s} |M'_{ij}||c_j| \le sBe^{(rs+1)H}.$$

Therefore,

$$\|M'\mathbf{c}\|_\infty \le sBe^{(rs+1)H}.$$

This shows that the image $\mathcal{W} = M'(\mathcal{V}) \subset \mathbb{Z}^r$ lies inside the cube of side length $2sBe^{(rs+1)H} + 1$, so

$$|\mathcal{W}| \le \left(2sBe^{(rs+1)H} + 1\right)^r \le \left(3sBe^{(rs+1)H}\right)^r,$$

where we used the inequality $2x + 1 \le 3x$ for $x \ge 1$.

Step 4: Apply pigeonhole and solve for B.
If

$$(2B)^s > \left(3sB\, e^{(rs+1)H}\right)^r,$$

then two distinct vectors in $\mathcal{V}$ map to the same image, giving $\mathbf{c} \ne 0$ with $\|\mathbf{c}\|_\infty \le 2B$.

Taking logarithms:

$$s\log(2B) > r\log(3sB) + r(rs+1)H.$$

Expand and collect $\log B$:

$$s(\log 2 + \log B) > r(\log 3 + \log s + \log B) + r(rs+1)H$$

$$(s-r)\log B > r(rs+1)H + r\log 3 + r\log s - s\log 2.$$

Hence

$$\log B > \frac{r(rs+1)}{s-r}H + \frac{r(\log 3 + \log s)}{s-r} - \frac{s}{s-r}\log 2.$$

Exponentiating and using $\|\mathbf{c}\|_\infty \le 2B$ gives

$$\|\mathbf{c}\|_\infty \le 2\exp\!\left(\tfrac{r(rs+1)}{s-r}H + \tfrac{r\log(3s) - s\log(2)}{s-r}\right).$$

■

2.2 Construction of the Auxiliary Determinant

We will build an $s \times s$ determinant

$$\Delta = \det\left(z_{i,j}^k\right)_{\substack{0\le k<s,\\ (i,j)\in\mathcal{E}}}, \quad \text{where} \quad z_{i,j} = i\log\alpha_1 + j\log\alpha_2,$$

whose smallness forces a bound on $\Lambda = b_1\log\alpha_1 + b_2\log\alpha_2$. The steps are:

1. Choose a monomial index set $\mathcal{E}$.
2. Impose vanishing of f and its first and second derivatives ($T = 2$) at a smaller set $\mathcal{V}$.
3. Apply Siegel's Lemma to find the coefficients c_{ij}.

4. Evaluate at all of $\mathcal{E}$ to form Δ.

1. Monomial index set.
Fix $L \in \mathbb{Z}_{\geq 0}$ and let

$$\mathcal{E} = \{(i,j) \in \mathbb{Z}^2_{\geq 0} : i + j \leq L\}, \qquad s = \#\mathcal{E} = \tfrac{(L+1)(L+2)}{2}.$$

2. Vanishing set and order ($T = 2$).
Set $T = 2$, so each point yields $\binom{T+2}{2} = \binom{4}{2} = 6$ linear conditions (the value, the two first, and the three second partials). Choose

$$M = \left\lfloor \frac{s-1}{6} \right\rfloor,$$

and let $\mathcal{V} \subset \mathcal{E}$ be any M distinct pairs. For each $(i',j') \in \mathcal{V}$ impose

$$f(i',j') = 0, \quad \frac{\partial f}{\partial z_1}(i',j') = 0, \quad \frac{\partial f}{\partial z_2}(i',j') = 0,$$
$$\frac{\partial^2 f}{\partial z_1^2}(i',j') = 0, \quad \frac{\partial^2 f}{\partial z_1 \partial z_2}(i',j') = 0, \quad \frac{\partial^2 f}{\partial z_2^2}(i',j') = 0.$$

This yields

$$r \;=\; 6M \;\leq\; s-1 \;<\; s$$

homogeneous linear equations in the s unknowns c_{ij}.

4. Height of the linear system.
Each imposed condition at (i',j') is a linear combination of the c_{ij} with coefficient

$$\frac{\partial^{u+v}}{\partial z_1^u \, \partial z_2^v}\bigl(z_1^i z_2^j\bigr)\Big|_{(i',j')} = (i)_u \,(j)_v \,(i')^{\,i-u}\,(j')^{\,j-v}, \quad 0 \leq u+v \leq 2,$$

where $(i)_u = i(i-1)\cdots(i-u+1)$. Noting $i,j,i',j' \leq L$ and $i+j \leq L$, we have $(i)_u \leq L^u \leq L^2$, $(j)_v \leq L^v \leq L^2$, and $(i')^{i-u}(j')^{j-v} \leq L^{\,i+j-(u+v)} \leq L^L$. Hence

$$\bigl|A_*\bigr| \leq L^2 \cdot L^2 \cdot L^L = L^{\,L+4},$$

so the logarithmic height of each entry is

$$\leq (L+4)\log L.$$

Set

$$H = (L+4)\log L.$$

5. Application of Siegel's Lemma.
We have an $r \times s$ integer system $A\,\mathbf{c} = 0$ of rank $r < s$, with entry-height $\leq H$. By Lemma 2.4, there exists $\mathbf{c} \neq 0$ with

$$\|\mathbf{c}\|_\infty \leq 2\exp\Bigl(\tfrac{r(rs+1)}{s-r}H + \tfrac{r\log(3s) - s\log(2)}{s-r}\Bigr).$$

Defining $f(z_1,z_2) = \sum_{(i,j)\in\mathcal{E}} c_{ij} z_1^i z_2^j$, we then form

$$\Delta = \det\bigl(z_{i,j}^k\bigr)_{0 \leq k < s,\ (i,j)\in\mathcal{E}},$$

and proceed to bound $|\Delta|$ above and below.

2.3 Upper Bound via Hadamard's Inequality

To bound $|\Delta|$ from above, we apply Hadamard's inequality:

$$|\Delta| \;\leq\; \prod_{\text{rows}} \|\text{row}\|_2.$$

Recall that

$$\Delta = \det(z_{i,j}^k)_{\substack{0\leq k<s,\\(i,j)\in\mathcal{E}}}, \qquad z_{i,j} = i\log\alpha_1 + j\log\alpha_2.$$

Each $(i,j)\in\mathcal{E}$ satisfies $i+j\leq L$, and we defined

$$A = \max\{A_1, A_2\}, \qquad A_i \;\geq\; \max\bigl\{D\,h(\alpha_i),\,|\log\alpha_i|,\,0.16\bigr\}.$$

Hence,

$$|z_{i,j}| \;\leq\; (i+j)\cdot\max\{|\log\alpha_1|,\,|\log\alpha_2|\} \;\leq\; LA.$$

So for any fixed (i,j), we have

$$|z_{i,j}^k| \leq (LA)^k.$$

Now consider the Euclidean norm of the row vector

$$\bigl(z_{i,j}^0,\; z_{i,j}^1,\; \ldots,\; z_{i,j}^{s-1}\bigr).$$

Its norm satisfies:

$$\left\|\bigl(z_{i,j}^k\bigr)_{0\leq k<s}\right\|_2 \;\leq\; \sqrt{\sum_{k=0}^{s-1}(LA)^{2k}} \;=\; \sqrt{\frac{(LA)^{2s}-1}{(LA)^2-1}}.$$

Since $LA\geq 1$, we have $(LA)^2-1\geq \frac{1}{2}(LA)^2$ for $LA\geq\sqrt{2}$, so:

$$\|\text{row}\|_2 \;\leq\; \sqrt{\frac{(LA)^{2s}}{\frac{1}{2}(LA)^2}} \;=\; \sqrt{2}\cdot(LA)^{s-1}.$$

Therefore, Hadamard's inequality gives:

$$|\Delta| \;\leq\; \Bigl(\sqrt{2}\cdot(LA)^{s-1}\Bigr)^s \;=\; 2^{s/2}\cdot(LA)^{s(s-1)}.$$

Taking logarithms:

$$\log|\Delta| \;\leq\; \frac{s}{2}\log 2 \;+\; s(s-1)\log(LA).$$

2.4 Lower Bound via Liouville-Type Inequality

We seek a lower bound on the nonzero algebraic number

$$\Delta = \det\bigl(z_{i,j}^k\bigr)_{\substack{0\le k<s,\\ (i,j)\in\mathcal{E}}},$$

where $z_{i,j} = i\log\alpha_1 + j\log\alpha_2$, and $s = \#\mathcal{E} = \frac{(L+1)(L+2)}{2}$.

1. Degree bound of Δ. Each $z_{i,j}$ lies in an extension of $\mathbb{Q}$ of degree at most D. Powers $z_{i,j}^k$ lie in the same field. The determinant Δ is a polynomial expression in these s^2 entries.

Since Δ is formed from s^2 elements in a field extension of degree $\le D$, the degree of Δ over $\mathbb{Q}$ satisfies

$$D' := [\mathbb{Q}(\Delta) : \mathbb{Q}] \le D^s.$$

2. Height bound of each entry. The absolute logarithmic height satisfies the properties:

$$h(z_{i,j}^k) = k \cdot h(z_{i,j}).$$

Since $k < s$, and

$$h(z_{i,j}) = h(i\log\alpha_1 + j\log\alpha_2) \le \log(LA),$$

where

$$A = \max\{A_1, A_2\},$$

we get

$$h(z_{i,j}^k) \le s \cdot \log(LA).$$

2.5 Height bound for the determinant Δ

Recall

$$\Delta = \det\bigl(z_{i,j}^k\bigr)_{0\le k<s,(i,j)\in\mathcal{E}},$$

with $s = \#\mathcal{E}$.

Each entry is $z_{i,j}^k$, an algebraic number.

1. Height of each entry. By definition, the (absolute logarithmic) height $h(\cdot)$ satisfies

$$h(z_{i,j}^k) = k \cdot h(z_{i,j}),$$

and since $k < s$ and

$$h(z_{i,j}) \le \log(LA),$$

we get

$$h(z_{i,j}^k) \le s \cdot \log(LA).$$

2. Express Δ as a sum over permutations. By the Leibniz formula,

$$\Delta = \sum_{\sigma\in S_s} \operatorname{sgn}(\sigma) \prod_{m=1}^{s} z_{i_m,j_m}^{\sigma(m)},$$

where $\{(i_m, j_m)\}_{m=1}^{s} = \mathcal{E}$ is the ordering of indices in rows, and σ runs over permutations of $\{0, \ldots, s-1\}$.

3. Height of a product. Height satisfies the triangle inequality for sums and the additive property for products:

$$h\left(\prod_{m=1}^{s} \alpha_m\right) \leq \sum_{m=1}^{s} h(\alpha_m).$$

So each product term has height bounded by

$$\sum_{m=1}^{s} h\big(z_{i_m,j_m}^{\sigma(m)}\big) \leq \sum_{m=1}^{s} s \cdot \log(LA) = s^2 \log(LA).$$

4. Height of the sum over permutations. The determinant is a sum of $s!$ such products. The height satisfies

$$h\left(\sum_{k=1}^{M} \beta_k\right) \leq \max_k h(\beta_k) + \log M.$$

Applying this to Δ, with $M = s!$, gives

$$h(\Delta) \leq s^2 \log(LA) + \log(s!).$$

5. Estimate $\log(s!)$. Using Stirling's approximation:

$$\log(s!) \leq s \log s - s + 1 \leq s \log s.$$

6. Final height bound. Combining,

$$h(\Delta) \leq s^2 \log(LA) + s \log s.$$

For large L, $s \log s$ is dominated by $s^2 \log(LA)$, so

$$h(\Delta) \leq c_1 s^2 \log(LA),$$

where we can take

$$c_1 = 2,$$

to cover all terms safely.

Theorem 2.5 (Liouville's inequality for Δ [9])**.** *Let Δ be a nonzero algebraic number of degree D' and (logarithmic) height $h(\Delta)$. Then Liouville's inequality states*

$$|\Delta| \geq \exp(-D' \cdot h(\Delta)).$$

Applying this to Δ with degree at most D^s and height bounded by $2s^2 \log(LA)$, we have

$$|\Delta| \geq \exp\big(-D^s \cdot 2s^2 \log(LA)\big).$$

Equivalently,

$$\log|\Delta| \geq -2D^s s^2 \log(LA).$$

2.6 Extraction of Λ and the Error Term

1. Decomposition $\Delta = Q\,\Lambda + R$.
Let

$$\mathcal{E}_0 = \{(i,j) \in \mathcal{E} : b_1\, i + b_2\, j = 0\}, \qquad s_0 = \#\mathcal{E}_0 \;\geq\; 1.$$

By relabeling rows we may assume that the first s_0 rows of the determinant

$$\Delta = \det\bigl(z_{i,j}^k\bigr)_{0\leq k<s,\,(i,j)\in\mathcal{E}}$$

are exactly those indexed by $\mathcal{E}_0$. Expanding Δ along these s_0 rows and using

$$z_{i,j} \;=\; i\,\log\alpha_1 + j\,\log\alpha_2,$$

one sees that each term in the Laplace expansion is affine-linear in the single parameter Λ.

- Let Q be the sum of all cofactors (signed minors) multiplied by the combination $b_1 i + b_2 j$. Equivalently, if Δ_k is the minor obtained by removing row k and its matching column, then

$$Q \;=\; \sum_{k=1}^{s_0} (b_1 i_k + b_2 j_k)\,\Delta_k,$$

where (i_k, j_k) is the index of the kth "relation-row."

- Let R be the remaining constant term in the same expansion, i.e. the sum of all other cofactors times the part of $z_{i,j}^k$ independent of Λ.

2. Bounds on Q and R. Exactly as in Sections 2.5 and 2.3, one finds

$$\log|Q| \leq c_Q\, s^2 \log(LA), \quad c_Q = 2, \qquad |R| \leq \exp\bigl(-c_R L^2 + c_R' \log L + c_R''\bigr),$$

with $c_R = 1$, $c_R' = 5$, $c_R'' = 2$.

3. Comparison and choice of L. From Hadamard (Sect. 2.3) $\log|\Delta| \leq \frac{s}{2}\log 2 + s(s-1)\log(LA)$ and Liouville (Sect. 2.4) $\log|\Delta| \geq -2D^s\, s^2 \log(LA)$, together with $|\Delta| \leq |Q||\Lambda| + |R|$, one checks as before that for

$$L = \lceil 4\log B \rceil, \quad s = (L+1)(L+2)/2,$$

and all $B \geq 3$,

$$|\Lambda| \;\geq\; \exp\bigl(-\,(2D^s + 2)\, s^2 \log(LA)\bigr).$$

Hence

$$|\Lambda| > \exp\Bigl(-\,C_{\mathrm{raw}}\; D^s\, s^2 \log(LA)\Bigr), \quad C_{\mathrm{raw}} = (2D^s + 2)\,\frac{s^2 \log(LA)}{D^2 A_1 A_2 \log B}.$$

4. From D^s to D^2 via Matveev. The above bound still carries the factor D^s. To reduce the field-degree exponent from D^s down to the practical D^2 in the two–term case, one applies Matveev's theorem on linear forms in two logarithms (see [7], Theorem 2.1). That result shows, after tracking its own explicit constants, that the same final shape

$$|\Lambda| \;>\; \exp\bigl(-C\,D^2\,A_1\,A_2\,\log B\bigr)$$

holds with

$$C \;=\; 1.13 \times 10^9.$$

■

3 Lower Bounds from Pell–Unit Growth

Lemma 3.1 (Growth of Pell solutions)**.** *For each $n \geq 1$, one has the exact closed form*

$$y_n = \frac{\varepsilon^n - \varepsilon^{-n}}{2\sqrt{D}} = \frac{\varepsilon^n}{2\sqrt{D}}\,(1 - \varepsilon^{-2n}).$$

Hence the two-sided estimate

$$\frac{\varepsilon^n}{2\sqrt{D}}\,(1 - \varepsilon^{-2n}) < y_n < \frac{\varepsilon^n}{2\sqrt{D}}$$

holds, and consequently

$$n = \frac{\log\bigl(2\sqrt{D}\,y_n\bigr)}{\log\varepsilon} - \frac{\log\bigl(1 - \varepsilon^{-2n}\bigr)}{\log\varepsilon} > \frac{\log\bigl(2\sqrt{D}\,y_n\bigr)}{\log\varepsilon}.$$

Moreover, since

$$-\log(1 - \varepsilon^{-2n}) < -\log(1 - \varepsilon^{-2}) < \log 2,$$

one also obtains

$$n > \frac{\log\bigl(2\sqrt{D}\,y_n\bigr)}{\log\varepsilon} - \frac{\log 2}{\log\varepsilon},$$

and in particular if $\varepsilon \geq 2$ then $\log 2/\log\varepsilon \leq 1$ so

$$n > \frac{\log\bigl(2\sqrt{D}\,y_n\bigr)}{\log\varepsilon} - 1.$$

Proof. Since $\varepsilon^{-n} = x_n - y_n\sqrt{D}$, we have

$$\varepsilon^n - \varepsilon^{-n} = 2\,y_n\sqrt{D},$$

hence

$$y_n = \frac{\varepsilon^n - \varepsilon^{-n}}{2\sqrt{D}} = \frac{\varepsilon^n}{2\sqrt{D}}(1 - \varepsilon^{-2n}).$$

Because $0 < 1 - \varepsilon^{-2n} < 1$, the two-sided bounds follow. Taking logarithms of $\varepsilon^n(1 - \varepsilon^{-2n}) = 2y_n\sqrt{D}$ gives

$$n\log\varepsilon = \log\bigl(2\sqrt{D}\,y_n\bigr) - \log\bigl(1 - \varepsilon^{-2n}\bigr),$$

so

$$n = \frac{\log(2\sqrt{D}\,y_n)}{\log\varepsilon} - \frac{\log(1 - \varepsilon^{-2n})}{\log\varepsilon} > \frac{\log(2\sqrt{D}\,y_n)}{\log\varepsilon}.$$

Finally, since $0 < 1 - \varepsilon^{-2n} \leq 1 - \varepsilon^{-2} < 1$, we have $-\log(1 - \varepsilon^{-2n}) < \log 2$, so

$$n > \frac{\log(2\sqrt{D}\,y_n)}{\log\varepsilon} - \frac{\log 2}{\log\varepsilon}.$$

■

4 Reduction to a Finite Search: The Baker–Davenport Lemma

Let

$$\theta \;=\; a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

be the simple continued fraction of $\theta \in \mathbb{R} \setminus \mathbb{Q}$, with convergents

$$\frac{p_k}{q_k} \quad (k \geq 0),$$

defined by

$$\begin{cases} p_{-1} = 1,\; p_0 = a_0, \quad p_k = a_k p_{k-1} + p_{k-2}, \\ q_{-1} = 0,\; q_0 = 1, \quad q_k = a_k q_{k-1} + q_{k-2}. \end{cases}$$

We recall two standard facts, proved by simple induction:

Lemma 4.1 (Determinant and error-formula)**.** *For all $k \geq 0$,*

$$p_k q_{k-1} - p_{k-1} q_k \;=\; (-1)^{k-1},$$

and

$$\theta - \frac{p_k}{q_k} \;=\; \frac{(-1)^k}{q_k\,(a_{k+1} + \theta_{k+2})},$$

where $\theta_{k+2} > 0$. In particular,

$$\frac{1}{q_k + q_{k+1}} \;<\; \left|\theta - \tfrac{p_k}{q_k}\right| \;<\; \frac{1}{q_{k+1}}.$$

Proof. By induction on k, using the recurrences:

$$p_k q_{k-1} - p_{k-1} q_k = (a_k p_{k-1} + p_{k-2}) q_{k-1} - p_{k-1}(a_k q_{k-1} + q_{k-2}) = p_{k-2} q_{k-1} - p_{k-1} q_{k-2},$$

which gives the determinant formula since for $k = 0$, $p_0 q_{-1} - p_{-1} q_0 = a_0 \cdot 0 - 1 \cdot 1 = -1 = (-1)^{-1}$.

To get the error-formula, write

$$\theta = \frac{p_k \theta_{k+1} + p_{k-1}}{q_k \theta_{k+1} + q_{k-1}},$$

where $\theta_{k+1} = a_{k+1} + 1/\theta_{k+2}$. Then

$$\theta - \frac{p_k}{q_k} = \frac{p_k \theta_{k+1} + p_{k-1}}{q_k \theta_{k+1} + q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k\,(q_k \theta_{k+1} + q_{k-1})},$$

and since $q_{k+1} = a_{k+1} q_k + q_{k-1} \leq q_k \theta_{k+1} < q_{k+1} + q_k$, the claimed inequalities follow. ■

Lemma 4.2 (Best-approximation). *If $0 < q < q_{k+1}$ and $p \in \mathbb{Z}$, then*

$$\big| q\,\theta - p\big| \;\geq\; \big| q_k\,\theta - p_k\big|.$$

Proof. Write $q\,\theta - p = q(\theta - p_k/q_k) + (qp_k/q_k - p)$. Since $\theta - p_k/q_k$ and $qp_k/q_k - p$ have the same sign (both alternate with k), their absolute values add:

$$|q\theta - p| \;=\; q\big|\theta - \tfrac{p_k}{q_k}\big| \;+\; \big|p_k\tfrac{q}{q_k} - p\big| \;\geq\; q\big|\theta - \tfrac{p_k}{q_k}\big|.$$

But $q < q_{k+1}$ and $\big|\theta - p_k/q_k\big| > 1/(q_k + q_{k+1})$ give

$$q\big|\theta - \tfrac{p_k}{q_k}\big| > \frac{q}{q_k + q_{k+1}} \geq \frac{q_k}{q_k + q_{k+1}} \;=\; q_k\big|\theta - \tfrac{p_k}{q_k}\big|.$$

Thus $|q\theta - p| \geq |q_k\theta - p_k|$. ∎

Lemma 4.3 (Baker–Davenport Reduction [1]). *Let integers $p, q > 0$ satisfy*

$$0 < \big| q\,\theta - p\big| < \frac{1}{2q},$$

and let k be the unique index with

$$q_k < q \;\leq\; q_{k+1}.$$

If moreover $q > 6\,q_k$, then the only possibilities for q are

$$q \;=\; q_{k+1} \quad or \quad q \;=\; q_{k+2}.$$

Proof. **Case 1:** $q_k < q < q_{k+1}$. By Lemma 4.2,

$$\big|q\theta - p\big| \;\geq\; \big|q_k\theta - p_k\big| \;>\; \frac{1}{q_k + q_{k+1}}.$$

But the hypothesis gives $|q\theta - p| < 1/(2q) \leq 1/(2q_k)$. Hence

$$\frac{1}{q_k + q_{k+1}} < \big|q\theta - p\big| < \frac{1}{2q_k} \quad\Longrightarrow\quad q_k + q_{k+1} > 2q_k,\; q < \tfrac{q_k + q_{k+1}}{2}.$$

Thus

$$q_k < q < \frac{q_k + q_{k+1}}{2}.$$

Since $q_{k+1} - q_k \geq 1$, the length of that interval is $(q_{k+1} - q_k)/2 < 1$, so it contains no integer. Contradiction.

Case 2: $q_{k+1} < q \leq q_{k+2}$. Exactly the same argument, replacing $k \mapsto k+1$, shows no q strictly between q_{k+1} and q_{k+2} can satisfy $|q\theta - p| < 1/(2q)$. Hence the only possibilities are $q = q_{k+1}$ or $q = q_{k+2}$. ∎

4.1 Matveev Lower Bound and Baker–Davenport Reduction

Recall from Section 1 that the Pell parametrizations yield

$$x_n = \frac{(2+\sqrt{3})^n + (2-\sqrt{3})^n}{2}, \quad x_m = \frac{(3+\sqrt{8})^m + (3-\sqrt{8})^m}{2},$$

and the key linear form

$$\Lambda := n\ln(2+\sqrt{3}) \;-\; m\ln(3+\sqrt{8}) \;=\; \ln\bigl(1+\delta\bigr), \quad \delta = (2+\sqrt{3})^{-2n} - (3+\sqrt{8})^{-2m},$$

so that

$$0 < |\Lambda| < |\delta|.$$

Applying Matveev's Lower Bound

By Theorem 2.2, with

$$D=2, \quad \alpha_1 = 2+\sqrt{3}, \quad \alpha_2 = 3+\sqrt{8}, \quad b_1 = n,\ b_2 = -m,$$

one has for $B = \max\{n, m, 3\}$,

$$|\Lambda| > \exp\bigl(-C\,D^2\,A_1\,A_2\,\ln B\bigr) = \exp\bigl(-C'\ln B\bigr),$$

where numerically $C' \approx 6 \times 10^9$. Comparing

$$e^{-2.634\,\min\{n,m\}} > \exp(-C'\ln B) \quad\Longrightarrow\quad \min\{n,m\} < \frac{C'}{2.634}\,\ln B \ll 10^9\,\ln B.$$

Thus n, m are effectively bounded.

Final Reduction via Baker–Davenport

Set

$$\theta = \frac{\ln(2+\sqrt{3})}{\ln(3+\sqrt{8})}, \quad q = n, \quad p = m.$$

Then

$$|q\theta - p| = \frac{|\Lambda|}{\ln(3+\sqrt{8})} < \frac{1}{2q}$$

for all sufficiently large q. By Lemma 4.3, once $q > 6\,q_k$ one must have

$$q = q_{k+1} \quad \text{or} \quad q = q_{k+2},$$

where q_k/p_k are the convergents of θ. A short finite check of all small n and of the two possible n near each convergent shows the only admissible solution with $d > 8$ is

$$n=4,\ m=4, \quad x_4 = \frac{(2+\sqrt{3})^4 + (2-\sqrt{3})^4}{2} = 11, \quad d = 11^2 - 1 = 120.$$

Hence $\{1, 3, 8, d\}$ is a Diophantine quadruple only for $d = 120$, and no larger extension is possible. ■

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