# Hadwiger-Nelson Problem

Qi Huang

July 14, 2025

# Table of Contents

# The Problem

The following question was stated by Edward Nelson in 1950:

**Problem (The Hadwiger-Nelson Problem)**

*How many ways are there to color the plane so that no two points that are unit distance apart have the same color?*

# Chromatic Number of the Plane

One can use graph theory to look at this problem:

**Definition (Chromatic Number of the Plane)**

The chromatic number of a graph $G$, $\chi(G)$, is the minimum number of colors required to color the vertices of $G$ such that no two vertices that share an edge have the same color.

**Definition**

Let $\Gamma(\mathbb{R}^2)$ be the graph whose set of vertices is the set of all points in 2D space, and whose set of edges is the set of all pairs of points $\{P, Q\}$ that are a distance 1 apart from each other.

# Chromatic Number of the Plane

Let's restate the problem using this terminology:

**Problem (Hadwiger-Nelson Problem Restatement)**

*Find the chromatic number* $\chi(\Gamma(\mathbb{R}^2))$ *of the graph* $\Gamma(\mathbb{R}^2)$.

We denote $\chi(\Gamma(\mathbb{R}^2))$ by $\chi(\mathbb{R}^2)$, as it is easier to write.

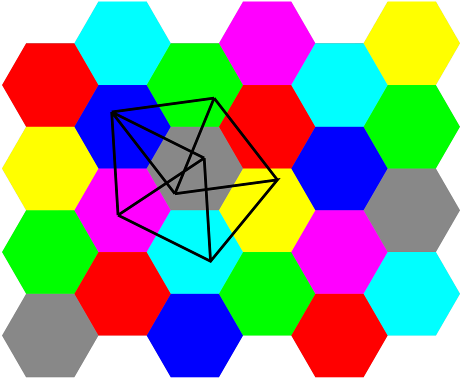
# De Bruijn-Erdös Compactness Theorem

**Theorem (De Bruijn-Erdös Compactness)**

*An infinite graph $G$ is $k$-colorable if and only if every finite subgraph of $G$ is $k$-colorable.*

**Theorem (Old Lower Bound)**

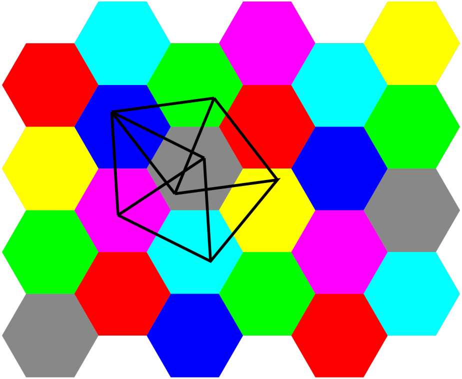
*The chromatic number of the plane* $\chi(\mathbb{R}^2)$ *satisfies* $\chi(\mathbb{R}^2) \geq 4$.



# Upper Bound

## Theorem (Upper Bound)

*The chromatic number of the plane is bounded from above, and* $\chi(\mathbb{R}^2) \leq 7$.



# New Lower Bound

**Theorem (2018 New Lower Bound)**

*The chromatic number of the plane* $\chi(\mathbb{R}^2)$ *satisfies* $\chi(\mathbb{R}^2) \geq 5$.

Aubrey de Grey (an amateur mathematician) constructed a graph with a chromatic number of 5 and proved a new lower bound.

# Proof of New 2018 Lower Bound

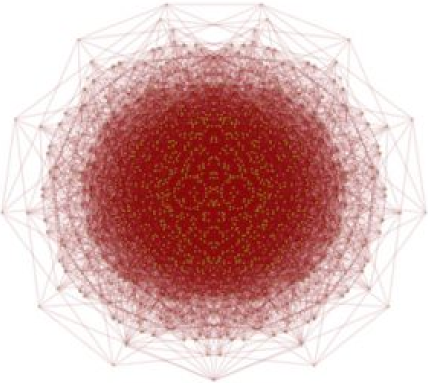


Figure: 1581-vertex graph

This graph can be verified by a computer to have chromatic number 5. There is no human-verifiable proof that this graph has a chromatic number of 5.

# Smallest Known Graph

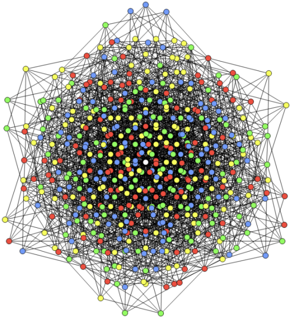


Figure: Jaan Part's 509-vertex graph

# 6-Coloring of the Plane

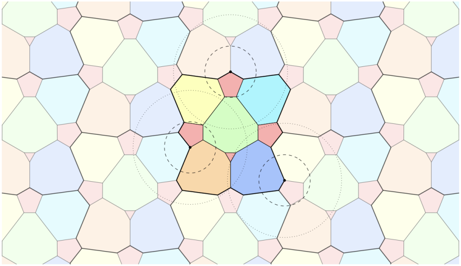


Figure: 6-coloring of a the plane.

There are no line segments unit distance apart for 5 colors and there are no line segments a distance of $d$ apart for the other color.

# Extensions of the Problem

We can extend the Hadwiger-Nelson problem to other fields, such as the fields $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, and $\mathbb{Q}(\sqrt{7})$.

# Thank you!