# THE HADWIGER-NELSON PROBLEM

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Abstract. In this paper, we will explore the Hadwiger-Nelson problem, a well-known question in graph coloring. Posed in 1950 by Edward Nelson, the problem asks for the minimum number of colors needed to color the plane so that no 2 points that are unit distance apart have the same color.

The best known lower bound on this number is 5, as there is a unit distance graph that could be embedded in the plane, cannot be colored with 4 colors, but can be colored with 5. The best known upper bound on the plane's chromatic number is 7, since there is a tiling of the plane with 7 colors which guarantees that no 2 points which are unit distance apart have the same color. The problem can be extended to $n$-dimensional Euclidean spaces and various fields.

## 1. Introduction

One may be familiar with the Four-Color Theorem, which states that any map can be colored using four or fewer colors. A similar problem in graph coloring involves finding the minimum number of colors required to color the plane such that no two points which are unit distance apart have the same color. This question is known as the Hadwiger-Nelson problem. A less formal way of thinking about the problem is to consider an arbitrary coloring of the plane. Now imagine a walk through the plane with steps of length 1. Each step of the walk must end at a point with a different color from the starting point of the step. The Hadwiger-Nelson problem asks for

*Date*: July 15, 2025.

the minimum number of colors we need to make a map of the plane such that the set of all walks we can imagine is the set of all possible walks with steps of length 1 on the 2-dimensional plane. From here, we will call any coloring of the plane where points unit distance apart are colored differently a "valid" coloring.

The problem can be rephrased in the language of graph theory. First, we consider all of the points in the plane as vertices of a graph called $\Gamma(\mathbb{R}^2)$, and line segments of length one connecting those points as edges of that graph. We use the idea of the chromatic number $\chi(\mathbb{R}^2) = \chi(\Gamma(\mathbb{R}^2))$, the minimum number of colors required to color a graph such that no two vertices which share an edge have the same color. The Hadwiger-Nelson problem asks for the chromatic number of the graph, or the minimum number of colors needed to color this graph. The current bounds on the chromatic number of the plane are $5 \leq \chi(\mathbb{R}^2) \leq 7$.

## 2. Preliminaries

We first give an informal definition of a graph. Generally speaking, a graph represents the relationships between a certain set of objects. These are the vertices of a graph, and the relationships between them are the graph's edges. The vertices of a graph can be anything, such as points, numbers, people, or colors. The edges of a graph can represent any relationship between those objects. For example, we can make a graph whose vertices represent the members of a group of people, such as Alice, Bob, and Charlie. The edges may represent whether there are friendships between them. For example, if Alice and Bob are friends, the graph may contain an edge between them.

Here is the formal definition of a graph:

**Definition 1.** *A graph is an ordered pair $G = (V, E)$, where $V$ is a set of objects known as vertices, and $E$ is a set of unordered pairs of vertices $\{u, v\}$.*

A common problem in graph theory is to find colorings of a graph's vertices such that two points adjacent to each other do not have the same color. We may ask for the minimum number of colors required for such colorings, which motivates us to define the chromatic number of a graph.

**Definition 2.** *The chromatic number $\chi(G)$ is the minimum number of colors needed to color $G$ such that no two vertices which share an edge have the same color.*

For the Hadwiger-Nelson problem, it will be convenient to define the following:

**Definition 3.** *Let $E$ be a non-zero commutative ring and $n \geq 1$ be an integer. We define $\Gamma(E^n)$ to be the graph whose set of vertices consists of all elements in $E^n$, and whose set of edges consists of all unordered pairs of vertices $\{v = (x_1, x_2, \ldots, x_n), v' = (x'_1, x'_2, \ldots, x'_n)\}$ for which*

$$(x'_1 - x_1)^2 + (x'_2 - x_2)^2 + \ldots + (x'_n - x_n)^2 = 1.$$

## 3. De Bruijn-Erdös Compactness Theorem

An infinite graph $G$ is $k$-colorable if and only if every finite subgraph of $G$ is $k$-colorable. In other words, $\chi(G)$ is the maximum chromatic number of all of the finite subgraphs of $G$.

Since $\Gamma(\mathbb{R}^2)$ is an infinite graph, and the set of its finite subgraphs is the set of all unit distance graphs in the plane, the maximum chromatic number of all unit distance graphs in the plane is equal to $\chi(\Gamma(\mathbb{R}^2))$ by the De Bruijn-Erdös Compactness Theorem. A way of approaching the Hadwiger-Nelson problem is trying to construct finite graphs with high chromatic numbers, which gives us lower bounds on $\chi(\mathbb{R}^2)$. However, no one has ever proven successfully that some finite unit distance graph has a greater chromatic number than all other finite unit distance subgraphs of the plane, which would solve the Hadwiger-Nelson problem.

By the de Bruijn-Erdös Theorem, if the plane is not 3-colorable, then there is a non-3-colorable graph embedded in $\Gamma(\mathbb{R}^2)$. We will prove the non-3-colorability of such a graph and show that the chromatic number of the plane is at least 4.

One could make an interesting connection between logic and the de Bruijn-Erdös Theorem. If one applies the compactness theorem of propositional calculus to the set of propositional variables "vertex $v$ has color $i$," and an infinite set of axioms stating that every vertex has exactly one color and adjacent vertices have different colors, one derives the de Bruijn-Erdös Theorem.

The Axiom of Choice is required to prove the de Bruijn-Erdös Theorem.

## 4. Classical Lower Bound

We may prove that the chromatic number of the plane is not 1. Consider the graph of two points (which are vertices) unit distance apart linked by an edge. If we try to color it with one color, then both vertices will have the same color and share an edge. This means the graph could only be colored with two colors.

We can also prove that the chromatic number of the plane is not 2. If we have a graph whose vertices are the vertices of an equilateral triangle with a side length of 1, and whose edges are the sides of the equilateral triangle, then it cannot be colored with two colors. Suppose the graph could be colored with two colors. Then we could use only red and blue to color it. If one of the vertices was colored red, then the other two vertices must be colored blue, since they are both connected to the red vertex. However, any 2 vertices of the graph must be colored differently from each other, since they are connected by an edge. This contradicts the fact that 2 vertices in the graph are colored red, so the graph cannot be colored with two colors.

Now, we will introduce a unit-distance graph which cannot be colored with 3 colors, showing that the chromatic number of the plane is at least 4.

**Theorem 4.1.** *The chromatic number of the plane* $\chi(\mathbb{R}^2)$ *satisfies* $\chi(\mathbb{R}^2) \geq 4$.

*Proof.* Let $A$, $B$, $C$, and $D$ be equilateral triangles of side length 1.

We place them in the plane such that $A$ and $B$ share a vertex, the pairs of triangles $A, C$ and $B, D$ share the edges of $A$ and $B$ opposite to that vertex, and the vertices of $C$ and $D$ opposite to their shared edges are unit distance apart.

Let $E$ be the vertex shared by $A$ and $B$.

Let $F$ be the vertex of $C$ opposite to the side of $C$ shared by $A$.

Let $L$ be the vertex of $D$ opposite to the side of $D$ shared by $B$.

Let $H$, $I$, $J$, and $K$ be the endpoints of the sides in $A$ and $B$ opposite to $E$, their shared vertex. We designate $H$ and $I$ to the vertices of $A$ and $J$ and $K$ to the vertices of $B$.

Let the graph $G$ be the graph whose vertices are all of the vertices of $A$, $B$, $C$, and $D$, and whose edges are all of the edges of $A$, $B$, $C$, and $D$, as well as the edge connecting vertices $F$ and $L$.

**Lemma 4.2.** $\chi(G) = 4$.

*Proof.* Assume for the sake of contradiction that the graph $G$ could be colored with 3 different colors, such as red, green, and blue. This means a valid 3-coloring of $G$ exists. Without loss of generality, we consider $E$ to be red red. Now, note that $H$ and $I$ must be colored blue and green in some order, since $H$ and $I$ must be colored differently from each other and both $H$ and $I$ must have a different color from $E$. Analogously, note that $J$ and $K$ must be colored blue and green in some order. Furthermore, note that $F$ must be red, as $H$ and $I$ are blue and green, and, similarly, $L$ must be red. Since $F$ and $L$ share an edge and are colored the same way, there is a contradiction to the assumption that a valid 3-coloring of $G$ exists and $G$ could be colored with three different colors. However, if we take the above coloring and change the color of $L$ to a 4th color yellow, then we obtain a valid coloring. This means that $G$ cannot be colored with 3 colors but can be colored with 4 colors, so $\chi(G) = 4$, and we are done. $\square$

Since $\chi(G) = 4$ by Lemma 4.2 and $G$ is embedded within $\Gamma(\mathbb{R}^2)$, we know that $\chi(\Gamma(\mathbb{R})^2) \geq 4$, and we are done.

$\square$

Note that the graph $G$ from the proof above is called the Moser spindle, which was discovered by Leo and William Moser soon after the Hadwiger-Nelson problem was posed. Before reading the proof, one may check for themselves that the graph $G$ is not 3-colorable.

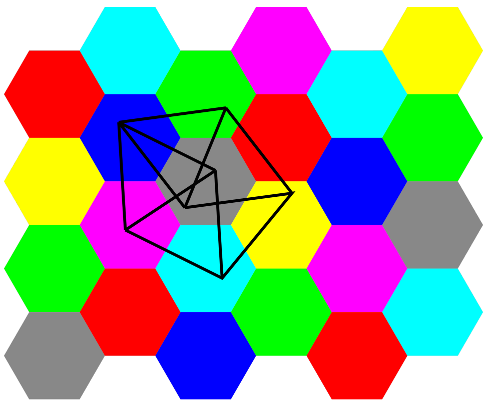

FIGURE 1. Moser Spindle and Heawood's Map

The Moser Spindle is not the only graph that requires that has a chromatic number of 4. Other unit distance graphs, such as the Golomb graph, are also 4-chromatic. In addition to the Moser spindle, the Golomb graph can be used to show that the chromatic number of the 2D-Euclidean plane is at least 4. The Golomb graph is shown below:

All line segments in Figure 2 have a length of 1, and Figure 2 cannot be colored with 3 distinct colors. Suppose the 3 colors we need are red, yellow,

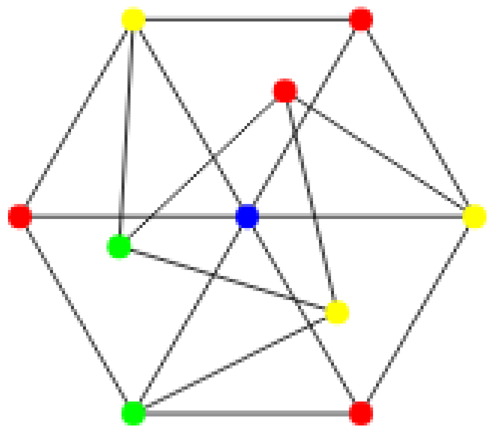

FIGURE 2. The Golomb Graph

and green. Note that if we use 3 different colors to color the vertices of the outer hexagon, then the middle vertex cannot be colored without having the same color as another vertex which shares an edge. Therefore, we must only use 2 colors to color the outer hexagon. Suppose the point at the right is colored yellow. Then we must color the vertices red, yellow, red, yellow, then red as we examine each one of them, going counterclockwise around the center vertex. Since every vertex in the central equilateral triangle shares an edge with a yellow vertex, they can only be colored red or green. However, one can show that a graph whose vertices are the vertices of an equilateral triangle, and whose edges are its sides cannot be colored with only 2 colors. Thus, it is impossible to color the Golomb graph with only 3 colors, and 4 are required. A valid coloring of the Golomb graph is shown in Figure 2.

## 5. Classical Upper Bound

**Theorem 5.1.** *The chromatic number of the plane is bounded from above, and* $\chi(\mathbb{R}^2) \leq 7$.

*Proof.* To prove this, we need to color the plane in such a way so that any two points unit distance apart have different colors.

Let the regular hexagons $A_i$ for $1 \leq i \leq 7$ be hexagons in the plane. Let the points $B_i$ be the vertices of $A_1$ for $1 \leq i \leq 6$. Let the regular hexagons $A_i$ for $2 \leq i \leq 7$ be arranged in the plane such that one of the edges of $A_i$ is $\overline{B_{i-1}B_i}$ for all values of $i$.

Now, we shift this union of hexagons so that they tile the entire plane. To make sure that no two points unit distance apart have the same color, we require that the diameter of each hexagon in the tiling is strictly less than 1 and strictly greater than $4\sqrt{3}/9 \approx 0.770$.

Since there is a tiling of the plane such that no two points unit distance apart have the same color, $\chi(\mathbb{R}^2) \leq 7$, and we are done.

$\square$

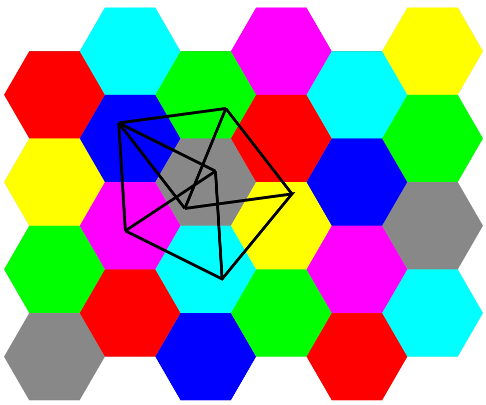

FIGURE 3. Another copy of Moser's Spindle and Heawood's Map (for convenience)

## 6. 2018 New Lower Bound

The chromatic number of the plane $\chi(\mathbb{R}^2)$ satisfies $\chi(\mathbb{R}^2) \geq 5$.

Aubrey de Grey (an amateur mathematician) constructed a graph with a chromatic number of 5 and proved a new lower bound.

De Grey is an anti-aging researcher whose work may help humans live to 1,000 years, and he solved the Hadwiger-Nelson problem while working on math in his spare time. He created a 5-chromatic unit distance graph by stitching together copies of the Moser Spindle, the 4-chromatic unit distance graph we used when proving $\chi(\mathbb{R}^2) \geq 4$. Although he initially obtained a monstrously large graph, he was able to create a smaller 5-chromatic graph with 1581 vertices. After de Grey's discovery, a Polymath project was started with the goal of creating smaller and smaller 5-chromatic unit distance graphs. The smallest known 5-chromatic unit distance graph has 509 vertices and was discovered by Jaan Parts.

In order to show that the chromatic number of any graph is 5, one could first prove that it is not 4-colorable, then show that it is 5-colorable. In his paper where he demonstrates that $\chi(\mathbb{R}^2) \geq 5$, de Grey outlined a computer algorithm that could verify that his graph was not 4-colorable. Interestingly, there is no human-verifiable proof that this graph has a chromatic number of 5.

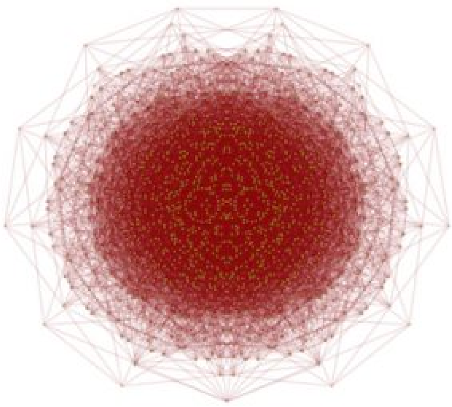

FIGURE 4. de Grey's 1581-vertex 5-chromatic graph

After de Grey published his 1581-vertex graph, mathematicians tried to find smaller unit distance graphs with a chromatic number of 5. The smallest known unit distance graph with a chromatic number of 5 was discovered by Jaan Parts, with 509 vertices.

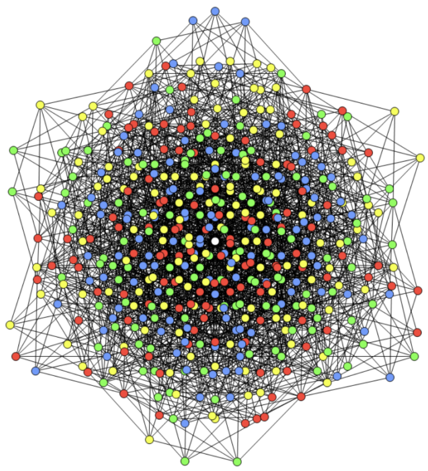

FIGURE 5. Jaan Part's 509-vertex 5-chromatic graph

## 7. Other Progress

Although there are no valid colorings of the plane with six colors, there are six-colorings of the plane satisfying the requirement where points unit distance apart are either differently colored or both a certain color $A$, and points a fixed distance $d$ apart that cannot both be $A$. The six-coloring below satisfies this requirement for all $d$ where $0.354 \leq d \leq 0.553$.

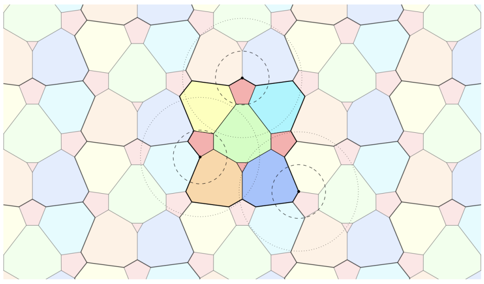

Figure 6. 6-coloring satisfying our modified condition for $0.354 \leq d \leq 0.553$

## 8. Extensions of the Problem

The problem can be extended to dimensions $n$ and number fields such as $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{7})$, and $\mathbb{Q}(\sqrt{3}, \sqrt{11})$. Essentially, $\mathbb{Q}(\sqrt{p})$ where $p$ is prime is defined as a number system supporting addition and multiplication with numbers of the form $a + b\sqrt{p}, a, b \in \mathbb{Q}$. Also, $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ is defined as a number system supporting multiplication and addition whose numbers can be written as $a + b\sqrt{p} + c\sqrt{q} + d\sqrt{pq}, a, b, c, d \in \mathbb{Q}$.

David A. Madore has proved that $\chi(\mathbb{Q}(\sqrt{2})^2) = 2$, $\chi(\mathbb{Q}(\sqrt{3})^2) = 3$, $\chi(\mathbb{Q}(\sqrt{7})^2) = 3$, and $4 \leq \chi(\mathbb{Q}(\sqrt{3}, \sqrt{11})^2) \leq 5$.

## 9. Conclusion and Acknowledgements

We conclude that the current known bounds for the chromatic number of the plane are $5 \leq \chi(\mathbb{R}^2) \leq 7$. The problem can also be extended to other fields and to $n$-dimensional space. Much progress had been made on the Hadwiger-Nelson problem, both by professional mathematicians and amateur math enthusiasts.

I would like to thank my instructor Simon Rubinstein-Salzedo and my TA Lucy Vuong for their feedback and guidance on this paper. This was written for an independent research and paper writing class at Euler Circle.

## 10. References

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