Graph Colorings

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Graph Theory Background

History

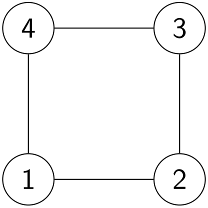
In 1852, Francis Guthrie was trying to color the counties in England such that any neighboring counties had different colors. Many people believed that every map could be colored with 4 colors. The first proof was given by Alfred Kempe in 1879 which was considered to be correct until a flaw was found 11 years later. The 4-color problem was only solved in 1976 and it was the first computer assisted proof.

What is a graph?

Definition 1.1

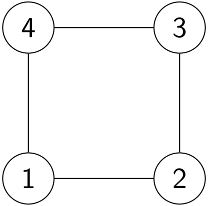
A graph is an ordered pair $G = (V, E)$ where V is a set of vertices and E is a set of edges such that each element in E is a two-element subset of V.

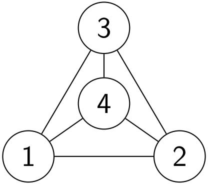
For example, let's say we have a graph
$G = ((1, 2, 3, 4), ((1, 2), (2, 3), (3, 4), (1, 4))$



K_n and C_n

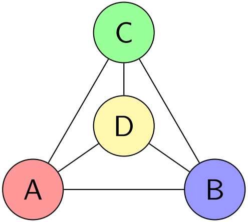
The graph K_n is the complete graph with n vertices. This means that every pair of vertices is connected with exactly one edge. The graph C_n is a cycle with n vertices.





What is a valid graph coloring?

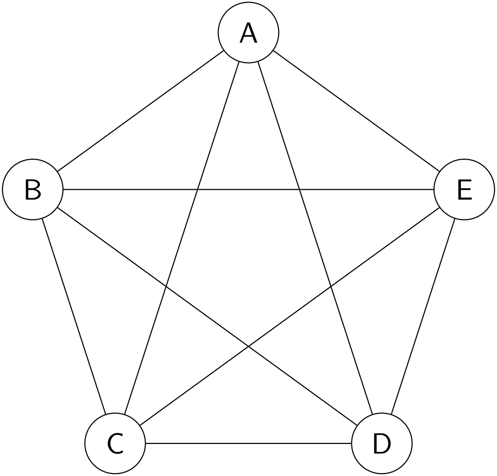
A valid graph coloring is an assignment of colors to vertices such that no two adjacent vertices share the same color.



The left graph illustrates a valid coloring, while the right graph is invalid because vertices A and D are adjacent and have been assigned the same color. The chromatic number of a graph is the minimum number of colors needed to color a graph.

What is a Planar graph?

A graph is planar if it can be drawn on the plane with no edge crossings.



Euler's Inequality for Planar graphs

Theorem 1.2 (Euler's Formula)

For any planar graph G if v is the number of vertices, e is the number of edges, and f is the number of faces then $v - e + f = 2$.

Now for a planar graph every face has at least 3 edges and every edge is in 2 faces. So we get $2e \geq 3f$. We can plug this into Euler's Formula to get $e \leq 3v - 6$ for all planar graphs.

Theorem 1.3 (Euler's Inequality for Planar graphs)

For any planar graph G if v is the number of vertices and e is the number of edges then $e \leq 3v - 6$.

Coloring planar graphs

The 6-color theorem

Theorem 2.1 (The 6-color theorem)

Every planar graph can be colored with 6 colors.

Since the degree of each vertex is the amount of edges going out from it, the sum of all the degrees is twice the number of edges (twice because each edge gets counted once for each vertex). So the average degree is $\frac{2e}{v}$. Using Euler's Inequality for Planar graphs we get $\frac{2e}{v} \leq 6 - \frac{12}{v}$. So the average degree is less than 6. This means there exists a vertex with degree 5 or less (let this be v').

The 6-color theorem

Theorem 2.2 (The 6-color theorem)

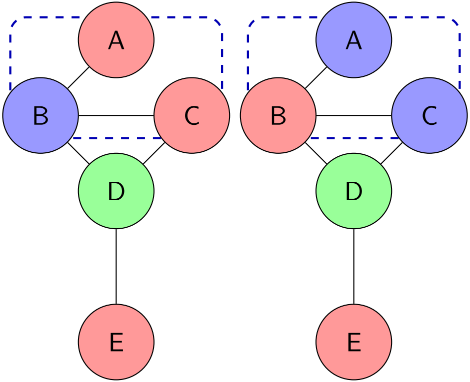
Every planar graph can be colored with 6 colors.

Next we use induction on the number of vertices.
Base case: When there are less than 6 vertices we can color all the vertices a different color.
Induction Step: When there are k vertices we assume that we can color our graph with 6 colors.
Induction Hypothesis: We aim to show that we can color any graph with $k+1$ vertices with 6 colors. Notice that we can color every vertex other than v' using 6 colors. Then, since v' has less than 6 neighbors we can assign it a 6^{th} color.
Therefore, every planar graph can be colored with 6 colors.

Kempe Chains

Definition 2.3

A Kempe chain is a maximally connected subgraph of a colored graph such that each node in the subgraph only uses one of 2 colors.



The 5-color theorem

Theorem 2.4 (The 5-color theorem)

Every planar graph can be colored with 5 *colors.*

From earlier we know that there exists a vertex v' that has degree at most 5. We can use induction on the amount of vertices.

Base case: When there are less than 5 vertices we can color all the vertices a different color.

Induction Step: When there are k vertices we assume that we can color our graph with 5 colors.

Induction Hypothesis: We aim to show that we can color any graph with $k + 1$ vertices with 5 colors. Notice that we can color every vertex other than v' using 5 colors. Then, if v' has less than 5 neighbors we can color it a 5^{th} color. If it has 5 neighbors but two colors repeat then we can still color it with a 5^{th} color.

The 5-color theorem

If it has 5 neighbors and all the neighboring vertices are different colors then we need to free up a color. Label the neighbors v_1, v_2, v_3, v_4, v_5 and colors $1, 2, 3, 4, 5$. Let v_n be colored with the n^{th} color. Consider the Kempe chain with colors 1 and 3 containing v_1. If this Kempe chain does not contain v_3 then we can swap the colors in the Kempe chain and free up a color. Now consider the Kempe chain with colors 2 and 4 containing v_2. If this Kempe chain does not contain v_4 then we can swap the colors in the Kempe chain and free up a color. Notice that we can free up one of the colors because otherwise that would contradict the planarity of the graph.

The 4-color theorem

Theorem 2.5 (The 4-color theorem)

Every planar graph can be colored with 4 colors.

This is the first theorem which had a computer assisted proof.

Coloring bi-planar graphs

What is a bi-planar graph?

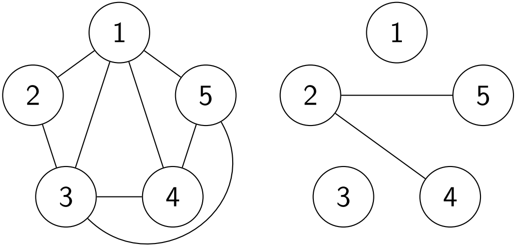
Definition 3.1

A bi-planar graph is a graph that can be decomposed into two planar graphs.

An example of a bi-planar graph

One example of a bi-planar graph is K_5. We know it is not planar because it does not satisfy $e \leq 3v - 6$.

An example of a bi-planar graph



The Earth-Moon Problem

Conjecture 3.2

Every bi-planar graph can be colored with 9 colors.

To this day the problem remains unsolved. However, there are bounds for the amount of colors needed.

What is a join of two graphs?

Definition 3.3

A join of two graphs G and H is represented by $G + H$. The vertex set in the join is $V_G \cup V_H$ and an edge (a, b) is in the edge set of the join if and only if any of the following are satisfied

- The edge $(a, b) \in E_G$
- The edge $(a, b) \in E_H$
- The vertex $a \in V_G$ and $b \in V_H$

Lower bound

The lower bound is 9 colors. This means that we need at least 9 colors to color each graph. An easy way to show this is to find an example of a graph that requires 9 colors. One graph is $K_6 + C_5$. This has chromatic number 9 because K_6 requires 6 colors and C_5 requires 3 colors.

Upper bound

The upper bound is 12 colors. This means every graph can be colored with 12 colors. To show this we first notice that in bi-planar graphs there are at most twice as many edges as in planar graphs. So we get

Theorem 3.4

For bi-planar graphs $e \leq 6v - 12$.

Using a similar method from earlier we can find the average degree of a bi-planar graph. We get $\frac{2e}{v} \leq 12 - \frac{24}{v}$. So the average degree is less than 12. Therefore there exists a vertex with degree 11 or less. Let this vertex be v'.

Upper bound

Next we use induction on the number of vertices.
Base case: When there are less than 12 vertices we can color all the vertices a different color.
Induction Step: When there are k vertices we assume that we can color our graph with 12 colors.
Induction Hypothesis: We aim to show that we can color any graph with $k+1$ vertices with 12 colors. Notice that we can color every vertex other than v' using 12 colors. Then, since v' has less than 12 neighbors we can assign it a 12^{th} color.
Therefore, every planar graph can be colored with 12 colors.

Candidate graphs

Next we will look at a graph which could change the bound for the Earth-Moon problem. First we need to understand how a strong product works.

What is a strong product?

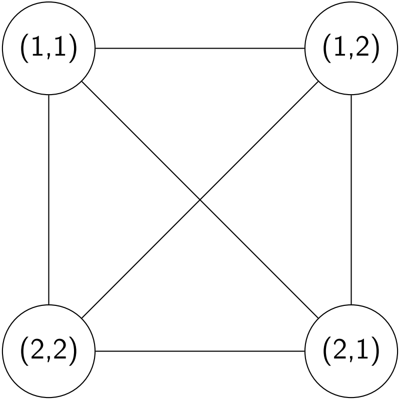
Definition 3.5

The strong product of two graphs G and H is written like $G \boxtimes H$. The vertex set of $G \boxtimes H$ is $V_G \times V_H$. Two vertices (g_1, h_1) and (g_2, h_2) are in the edge set of $G \boxtimes H$ if one of the following conditions is satisfied.

- $g_1 = g_2$ and $(h_1, h_2) \in E_H$
- $h_1 = h_2$ and $(g_1, g_2) \in E_G$
- $(g_1, g_2) \in E_G$ and $(h_1, h_2) \in E_H$

An example of a strong product

Let $V_G = (1, 2)$ and $E_G = ((1, 2))$. Let $V_H = (1, 2)$ and $E_H = ((1, 2))$. The the vertex set of the strong product is $((1, 1), (1, 2), (2, 1), (2, 2))$. And every pair of edges is connected.



Candidate graphs

Next we will look at a graph which could change the bound for the Earth-Moon problem. One graph is $C_5 \boxtimes K_4$ minus one vertex. We know that this graph has chromatic number 10. However, it remains unknown if the graph is bi-planar or not. There are 19 vertices and 99 edges. However, this problem is NP-Complete and the runtime algorithm is $O(2^e(v+e))$. So using brute-force to check if the graph is bi-planar would take many years.

Graphs of higher thickness

What is the thickness of a graph?

Definition 4.1

The thickness of a graph is the minimum number of planar graphs whose union is G.

For example all graphs with thickness 2 are bi-planar and all graphs with thickness 1 are planar.

Coloring graphs of thickness n

Naturally we ask how many colors does it take to colors graphs of thickness n.

Theorem 4.2

The maximum chromatic number of a graph with thickness n is $6n$.

Theorem 4.3

The minimum number of colors needed to color all graphs with thickness n is $6n - 2$ if $n \geq 3$.

Conclusion

Applications

- Search Engines
  - Use graphs to rank recommendations
- Airline Routes
  - Treat the airports as vertices and path the flights travel as edges
  - Helps minimize cost and time
- Social Networks
  - Influences recommendations

Conclusion

- Every planar graph can be colored with 4 colors.
- Every bi-planar graph can be colored with 12 colors.
- To color all bi-planar graphs at least 9 colors are required.
- All graphs with thickness n can be colored with $6n$ colors.
- For $n \geq 3$ we need at least $6n - 2$ colors to color all graphs with thickness n.