GRAPH COLORINGS

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1. Abstract

This paper surveys classical and extended results on vertex coloring in planar and nearly planar graphs. We begin with constructive proofs of the 6-color and 5-color theorems, then explore how these results extend to graphs of higher thickness—the minimum number of planar subgraphs into which a graph can be decomposed. We also introduce the concept of skewness, defined as the minimum number of edges whose removal yields a planar graph, and examine its effect on the chromatic number. We explore various types of graphs such as $k-$degenerate graphs and also edge colorings.

2. Introduction

The study of graph coloring originated from a simple question: how many colors are needed to color a map so that no two adjacent regions share the same color? In 1852, Francis Guthrie posed this question while trying to color a map of the counties of England. This led to the famous Four Color Problem, which conjectured that four colors are always sufficient to color any map.

A proposed proof by Alfred Kempe in 1879 was widely accepted for over a decade until a flaw was discovered in 1890. The problem remained unsolved for nearly a century, until Kenneth Appel and Wolfgang Haken finally proved the Four Color Theorem in 1976 using extensive computer assistance making it the first major theorem to be proven in such a way.

In graph theory, a vertex coloring is an assignment of colors to the vertices of a graph such that no two adjacent vertices share the same color. The chromatic number of a graph is the minimum number of colors needed for such a coloring. A graph is said to be planar if it can be drawn on the plane without any edges crossing.

3. Background

Definition 1. A graph $G = (V, E)$ is an ordered pair of two sets. The elements of the edge set are two-element subsets of the vertex set.

For this paper we will be coloring the vertices of the graph and not the edges.

Definition 2. A valid vertex coloring of a graph is an assignment of colors to each vertex such that any adjacent vertices are different colors.

Definition 3. A vertex v is a neighbor of v' if there is an edge between v and v'.

Definition 4. The chromatic number of a graph is denoted by $\chi(G)$ and is the minimum number of colors needed to color G.

Definition 5. The degree of a vertex is the number of neighbors it has.

Definition 6. A graph is regular if every vertex has the same degree.

Definition 7. A graph is bi-partite if the vertices can be separated into 2 disjoint sets X and Y such that every edge connects a vertex in X to a vertex in Y.

Definition 8. The bi-partite graph $K_{x,y}$ is the complete bi-partite graph where the vertex set is divided into two disjoint sets X and Y such that every vertex in X is adjacent to every vertex in Y. The sets X and Y have x and y vertices respectively.

4. Theorems

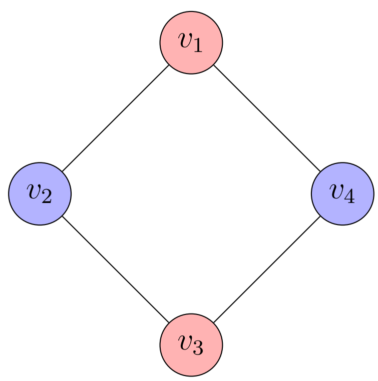
4.1. **Planar Graphs.** We let v be the number of vertices, f be the number of faces and e be the number of edges in a graph G.

Theorem 4.1 (Euler's formula)**.** *For every planar graph G, $v - e + f = 2$.*

Definition 9. We denote the graph with n vertices which is a cycle with C_n. We denote K_n to be the complete graph with n vertices.

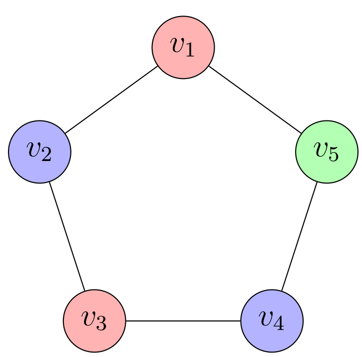
Theorem 4.2. *For all even n, $\chi(C_n) = 2$.*

Here is one way to color C_n with 2 colors when n is even. We color v_k red if k is odd and v_k blue if k is even.



Theorem 4.3. *For all odd n, $\chi(C_n) = 3$.*

Here is one way to color C_n with 3 colors when n is odd. We color v_k red if k is odd and v_k blue if k is even. However, since v_1 and v_n are adjacent we have to change the color of v_n. So we can color it green.

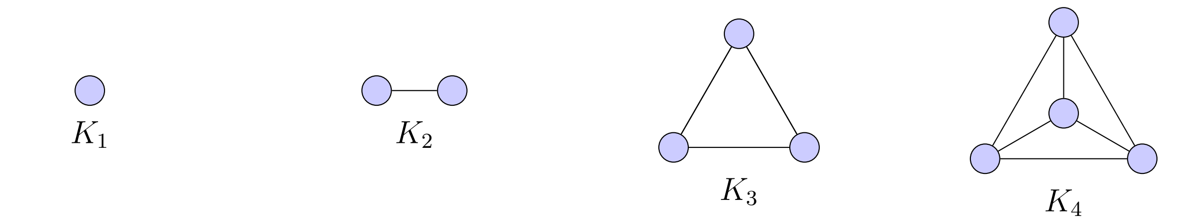


Theorem 4.4. *If G is a planar graph then $e \leq 3v - 6$.*

Proof. By Euler's formula we know that $e - f = v - 2$. Since each face has at least 3 edges and each edge is counted in 2 faces we know that $2e \leq 3f$. Plugging this into Euler's formula we get $e \leq 3v - 6$ as desired. ∎

Theorem 4.5. *The graph K_n is planar if and only if $n \leq 4$.*

Proof. First we will show that for $n \leq 4$ the graph K_n is planar.



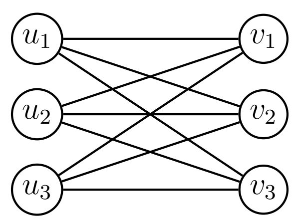
Next we will show that for $n > 4$ the graph K_n is not planar. Notice that in K_n there are $\frac{n(n-1)}{2}$ edges and n vertices. So in order for our graph to be planar it must satisfy $\frac{(n-1)n}{2} \leq 3n - 6$. So this means that $12 \leq 7n - n^2$. However, this is not satisfied for $n > 4$. Thus, K_n is planar if and only if $n \leq 4$. ∎

Theorem 4.6. *Every bi-partite graph can be colored with 2 colors.*

Proof. Since we can split a bi-partite graph into sets X and Y we can just color the vertices in X blue and the vertices in Y red. ∎

Theorem 4.7. *The graph $K_{3,3}$ is not planar.*

Proof. First notice that by Euler's formula we get $6 - 9 + f = 2$ so $f = 5$. This means that there are 5 faces. Since there cannot exist a triangle in a bi-partite graph it must be that each face has at least 4 edges. Since each edge is counted in exactly 2 faces, we need at least $\frac{5 \cdot 4}{2} = 10$ edges. However, we only have 10 edges. Therefore, $K_{3,3}$ is not planar since it doesn't satisfy Euler's formula.



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Theorem 4.8 (6-color theorem)**.** *Every planar graph G satisfies $\chi(G) \leq 6$.*

Proof. First notice that the sum of the degrees of all the vertices in any graph is $2e$ because each edge gets counted twice (once from each vertex). So, the average degree is $\frac{2e}{v}$. Since $e \leq 3v - 6$ we get $\frac{2e}{v} \leq 6 - \frac{12}{v}$. This means that the average degree is less than 6 so there exists a vertex with degree 5. We use induction on the number of vertices in our graph.

Base Case: $v \leq 6$. When there are fewer than 7 vertices we can color each vertex a different color so $\chi(G) \leq 6$.

Induction Hypothesis: $v = k$. We assume that any planar graph with k vertices can be colored with 6 or fewer colors.

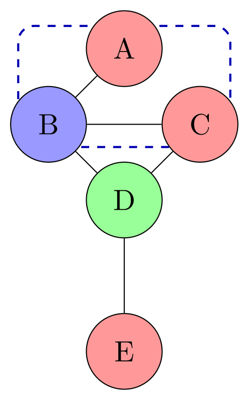
Induction Step: $v = k + 1$. We need to show that any planar graph with $k + 1$ vertices

has chromatic number less than 7. From earlier we know that there exists a vertex such that the degree is less than 6. Let this vertex be v'. We know that we can 6-color a graph with k vertices so we can color every other vertex with 6 or fewer colors. Then since v' has fewer than 6 neighbors it can be colored with the 6^{th} color.
Therefore, every planar graph G satisfies $\chi(G) \leq 6$. ■

Next we will show a stronger version of the $6-$color theorem. To show this we first define a Kempe Chain.

Definition 10. A Kempe chain is a maximally connected subgraph of a colored graph such that each node in the subgraph only uses one of 2 colors.

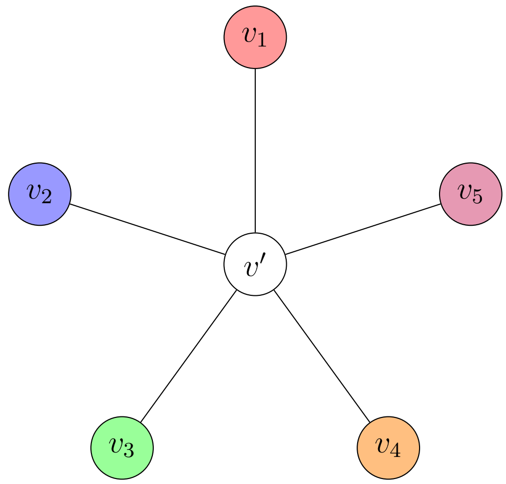
We can swap the colors in a Kempe chain and it will still be a valid coloring of the graph. One example of a Kempe chain is shown below.



Theorem 4.9 (5-color theorem)**.** *Every planar graph G satisfies $\chi(G) \leq 5$.*

Proof. We can use induction on the number of vertices.
Base Case: $v \leq 5$. We can color each vertex a different color so when $v \leq 5$ we can color the graph with 5 colors.
Induction Hypothesis: $v = k$. We assume that we can color any graph with k vertices using 5 colors.
Induction Step: $v = k + 1$. We aim to show that we can color any graph with $k + 1$ vertices. Let v' be the vertex with degree at most 5. We know that we can color all other vertices with 5 colors. If $deg(v') < 5$ then we can color v' with the 5^{th} color. If $deg(v') = 5$ but two neighboring vertices have the same color then we can still color v' with a 5^{th} color. If $deg(v') = 5$ and all the neighboring vertices are different colors then we need to free up a color. Label the neighbors v_1, v_2, v_3, v_4, v_5 and colors $1, 2, 3, 4, 5$. Let v_n be colored with the n^{th} color.

Consider the Kempe chain with colors 1 and 3 containing v_1. If this Kempe chain does not contain v_3 then we can swap the colors in the Kempe chain and free up a color. Now consider the Kempe chain with colors 2 and 4 containing v_4. If this Kempe chain does not contain v_4 then we can swap the colors in the Kempe chain and free up a color. Notice that we can free up one of the colors because otherwise that would contradict the planarity of the graph. So every planar graph is $5-$colorable.



■

Theorem 4.10 (4-color theorem)**.** *Every planar graph can be colored with 4 colors.*

Modern proofs of the 4−color check if certain un-avoidable configurations can be colored with 4 colors. The original proof by Kenneth Appel and Wolfgang Haken had checked 1482 configurations. Today this number has been further reduced to 633.

Definition 11. The skewness of a graph is the minimum amount of edges that need to be removed such that the graph is planar. We let $\mu(G)$ be the skewness of a graph G.

Theorem 4.11. *Let G be a graph such that $\mu(G) \leq 2$. Then $\chi(G) \leq 5$.*

Proof. First notice that $e - 2 \leq 3v - 6$ since removing 2 edges will keep the graph planar. This means that $e \leq 3v - 4$. Next define v_i to be the number of vertices in our graph with degree i. Then by minimality, $v_i = 0$ for $i < 5$. We also know that $\sum_i v_i = v$ and $\sum_i v_i \cdot i = 2e$. So we get $2e \leq 6v - 8$ which is equivalent to $8 \leq \sum_i (6-i)v_i = v_5 - v_7 - 2v_8 \ldots$. This means that $v_5 \geq 8$ so at least 8 vertices have degree 5. Now consider a graph G such that the removal of edges e and e' will result in the graph being planar. We can pick a vertex v' with degree 5 or less such that v' is not incident with the edges e or e'. If we delete and contract edges that are incident to v' we will get a new graph $\overline{G}$ which has skewness at most 2. Therefore, if $\mu(G) \leq 2$ then $\chi(G) \leq 5$. ■

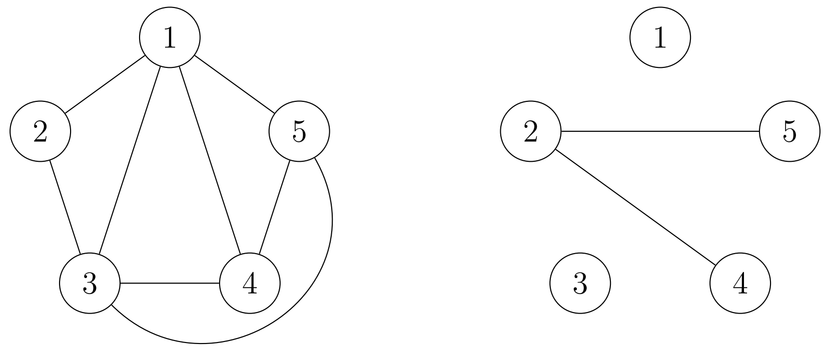
Theorem 4.12. *Let G be a graph such that $\mu(G) \leq 5$. Then $\chi(G) \leq 6$.*

Proof. First notice that $e \leq 3v - 1$. So letting v_i be the number of vertices in our graph with degree i we can find again that there exists at least one vertex with degree at most 5. We can let this degree be v. So $\mu(G - v) \leq \mu(G) \leq 5$ so there exists a valid 6-coloring of $G - v$ that extends to the graph G. ■

4.2. **Bi-planar Graphs.** We call a graph bi-planar if it can be decomposed into 2 planar graphs. The Earth-Moon problem (unsolved) asks how many colors are needed to color all bi-planar graphs. We will prove the lower bound and upper bound for the number of colors needed to color all bi-planar graph.

Theorem 4.13. *K_5 is bi-planar.*

Proof. Notice that K_5 has 10 edges and 5 vertices so it doesn't satisfy $e \leq 3v - 6$. This means it is not planar. Next notice that it is bi-planar because we can write it as the union of the following graphs.



Planar Subgraph with 8 Edges   Planar Subgraph with Remaining 2 Edges

■

Theorem 4.14. *If G is bi-planar then $e \leq 6v - 12$.*

Proof. By theorem 3.1 we know that $e \leq 3v - 6$ for all planar graphs. Since a bi-planar graph can be decomposed into 2 planar graphs it as at most twice as many edges has a planar graph. So for bi-planar graphs we know that $e \leq 6v - 12$. ■

Using this we can find an upper bound for the chromatic number of a bi-planar graph.

Theorem 4.15. *If G is bi-planar then $\chi(G) \leq 12$.*

Proof. Using theorem 3.5 we can find the average degree of our bi-planar graph. We get $\frac{2e}{v} \leq 12 - \frac{24}{v} < 12$. So the average degree is less than 12. This means there exists a vertex with degree 11 or less. Let this vertex be v'. Next we use induction on the number of vertices.
Base Case: $v \leq 12$. When there are fewer than 13 vertices we can color each vertex a different color so $\chi(G) \leq 12$.
Induction Hypothesis: $v = k$. We assume that any bi-planar graph with k vertices can be colored with 12 or fewer colors.
Induction Step: $v = k + 1$. We need to show that any graph with $k + 1$ vertices has chromatic number less than 13. We know that we can 12-color a graph with k vertices so we can color every vertex other than v' with 12 or fewer colors. Then since v' has fewer than 12 neighbors it can be colored with the 12^{th} color.

Therefore, every planar graph G satisfies $\chi(G) \leq 12$. ■

Definition 12. A join of two graphs G and H is represented by $G + H$. The vertex set in the join is $V_G \cup V_H$ and an edge (a, b) is in the edge set of the join if and only if any of the following are satisfied

- The edge $(a, b) \in E_G$
- The edge $(a, b) \in E_H$
- The vertex $a \in V_G$ and $b \in V_H$.

Theorem 4.16. *In order to color all bi-planar graphs at least* 9 *colors are needed.*

Proof. In order to show this we need to find an example of a bi-planar graph with chromatic number 9. One example $K_6 + C_5$. This has chromatic number 9 because K_6 requires 6 colors and C_5 requires 3 different colors. ∎

Definition 13. The strong product of two graphs G and H is written like $G \boxtimes H$. The vertex set of $G \boxtimes H$ is $V_G \times V_H$. Two vertices (g_1, h_1) and (g_2, h_2) are in the edge set of $G \boxtimes H$ if one of the following conditions is satisfied.

- $g_1 = g_2$ and $(h_1, h_2) \in E_H$
- $h_1 = h_2$ and $(g_1, g_2) \in E_G$
- $(g_1, g_2) \in E_G$ and $(h_1, h_2) \in E_H$

Researchers are working on shrinking the bounds for this problem. To increase the upper bound researchers are running programs to find bi-planar graphs that have chromatic number 10. To lower the upper bound researchers are using methods such as contraction and Kempe chains. One candidate for this problem is $K_4 \boxtimes C_5$. Researches have proved this graph has chromatic number 10. However, we can see that it is not bi-planar because it doesn't satisfy $e \leq 6v - 12$. For this graph removing a vertex keeps the chromatic number 10 so researches are trying to show whether or not the graph $K_4 \boxtimes C_5$ minus 1 vertex is bi-planar or not. We know that this graph has 19 edges and 99 vertices. However, since checking if a graph is bi-planar or not is NP-complete and the run time to check if a graph is bi-planar is $O(2^e(v+e))$ it is very hard to tell if this graph is bi-planar or not with brute force methods.

4.3. **Graphs of higher thickness.** A graph has thickness n if it can be decomposed into n planar graphs and cannot be decomposed into $n + 1$ planar graphs. So a bi-planar graph has thickness 2. We aim to find the number of colors needed to color a graph with thickness n.

Theorem 4.17. *If G has thickness n then $e \leq 3vn - 6n$*

Proof. By theorem 3.1 we know that $e \leq 3v - 6$ for all planar graphs. Since a graph with thickness n can be decomposed into n planar graphs it has at most n times as many edges as a planar graph. So for bi-planar graphs we know that $e \leq 3vn - 6n$. ∎

Theorem 4.18. *If G has thickness n then $\chi(G) \leq 6n$.*

Proof. Using theorem 3.7 we can find the average degree of our bi-planar graph. We get $\frac{2e}{v} \leq 6n - \frac{12n}{v} < 6n$. So the average degree is less than $6n$. This means there exists a vertex with degree $6n - 1$ or less. Let this vertex be v'. Next we use induction on the number of vertices.
Base Case: $v \leq 6n$. When there are fewer than $6n + 1$ vertices we can color each vertex a different color so $\chi(G) \leq 6n$.
Induction Hypothesis: $v = k$. We assume that any graph with k vertices can be colored with $6n$ or fewer colors.
Induction Step: $v = k + 1$. We need to show that any graph with $k + 1$ vertices has chromatic number less than $6n + 1$. From earlier we know that there exists a vertex such that the degree is less than $6n$. Let this vertex be v'. We know that we can 6 color a graph with k vertices so we can color every other vertex with $6n$ or fewer colors. Then since v' has fewer than $6n$ neighbors it can be colored with the $6n^{\text{th}}$ color.
Therefore, every planar graph G satisfies $\chi(G) \leq 6n$. ∎

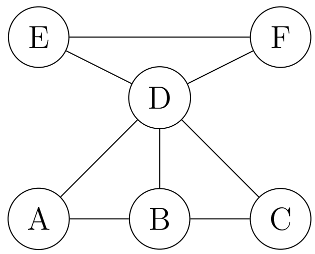
Theorem 4.19. *If G has thickness $n \geq 3$ then $\chi(G) \geq 6n - 2$.*

4.4. **Degenerate Graphs.**

Definition 14. A subgraph of a graph is formed by selecting a subset of the vertex set and edge set of the original graph.

Definition 15. A $k-$degenerate graph is a graph such that every subgraph has a vertex of degree at most k.

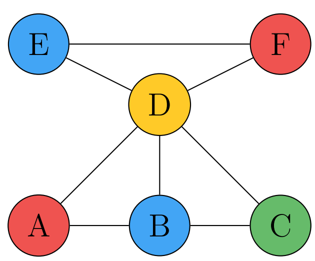
Here is an example of a $3-$degenerate graph.



Theorem 4.20. *Every $k-$degenerate graph can be colored with $k + 1$ colors.*

Proof. First we can remove a vertex in our graph G such that the vertex has degree at most k. Then since our new graph is a subgraph of G we can again remove a vertex with degree at most k. We can repeat this process until there are no vertices left. Then we can track the order in which we removed each vertex. We can color our vertices in reverse order. The last vertex we removed will be the first vertex we color and so on. Since every vertex has degree at most k at the time of its' removal we can assign it a $k + 1^{\text{th}}$ color due to the amount of neighbors it has. Therefore, we can color each vertex such that we only use $k + 1$ colors. So every $k-$degenerate graph has chromatic number at most $k + 1$. ■

This proof not only shows that we can color all $k-$degenerate graphs with $k + 1$ colors but also gives an algorithm to color each vertex. Here is a 4 coloring of the $3-$degenerate graph from earlier.



Theorem 4.21. *Planar graphs are $5-$degenerate.*

Proof. Notice that the average degree of a graph is less than 6 by Euler's inequality for planar graphs. This means that the graph has a vertex of degree at most 5. Since every subgraph of a planar graph is planar then every subgraph must have a vertex of degree at most 5. So all planar graphs as $5-$degenerate. ■

We use this theorem to again prove the $6-$color theorem

Theorem 4.22 (6-color theorem)**.** *Every planar graph can be colored with 6 colors.*

Proof. Since all planar graphs are $5-$degenerate and any $k-$degenerate graph can be colored with $k + 1$ colors, we can color any planar graph with 6 colors. ■

Next we will find an upper bound for the number of edges in a $k-$degenerate graph.

Theorem 4.23. *For every $k-$degenerate graph with v vertices, there is at most kv edges.*

Proof. Notice that we can always take out a vertex with degree at most k at any point. Every time we remove a vertex with degree at most k we are removing at most k edges. So we can keep removing vertices until there are none left and we removed v vertices and at most kv edges. So there are at most kv edges in the original graph. ■

4.5. **Edge colorings.**

Definition 16. An edge coloring of a graph is an assignment of colors to edges such that any edges that share a vertex are different colors.

Theorem 4.24 (Vizing's Theorem)**.** *If the maximum degree of a graph is Δ then the graph can be colored with either $\Delta + 1$ or Δ colors.*

Definition 17. If a graph requires $\Delta + 1$ vertices to be edge colored then we say it is class 2 otherwise it is class 1.

Definition 18. $\chi'(G)$ denotes the number of colors needed to color the edges of a graph G.

We will show an example of a class 1 and class 2 graph.

(a) Class 1: $\chi'(G) = \Delta = 3$ **(b)** Class 2: $\chi'(G) = \Delta + 1 = 3$

Theorem 4.25. *All bi-partite graphs are class* 1.

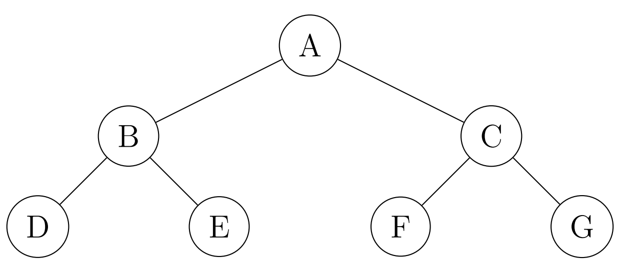
4.6. **Trees.**

Definition 19. A graph is maximally acyclic if adding a edge would make it not acyclic.

Definition 20. A graph is minimally connected if removing a edge would make it not connected.

Definition 21. A tree is a graph that is maximally acyclic and minimally connected.

Here is an example of a tree.



Theorem 4.26. *Every tree G has chromatic number at most* 2

Proof. Since every tree has no cycles, it has no cycles of odd length. Therefore every tree is bi-partite. So we can color our graph with 2 colors. ■

4.7. Graphs with max degree Δ.

Definition 22. Define the maximum degree of a graph with degree Δ

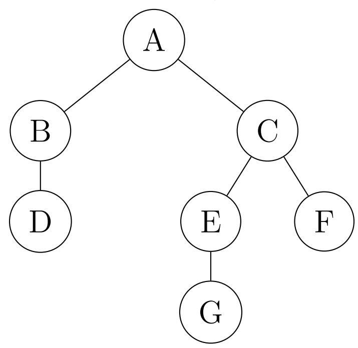
Theorem 4.27 (Brooks' Theorem)**.** *For a connected graph G that isn't complete and isn't an odd cycle it can always be colored with Δ colors.*

Before we look at the proof we first need to understand how the breadth-first search algorithm works.

Definition 23. The distance between two vertices v and v' is the number of edges in the shortest path between them.

Definition 24. Breadth-First Search (BFS) is a graph traversal algorithm that explores vertices in order of their distance from a chosen starting vertex. It begins at a root vertex v, visits all vertices at distance 1, then all vertices at distance 2, and so on, until every vertex has been visited.

Here is an example of BFS. If we start at the vertex A then we will visit vertices B and C. After that we visit vertices D, E, and F. Finally we visit vertex G.

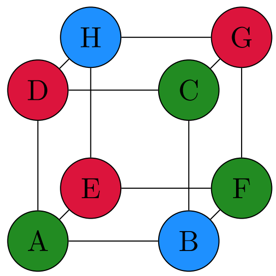


Proof of Brooks' Theorem. To prove this we will use casework based on if G is regular or not.
Case 1: Not Regular
If our graph is not regular then there exists a vertex with degree less than Δ. Let this vertex be v. Then we can apply a BFS starting at v. We can list the vertices in the order we visit each vertex. We color all the vertices in the reverse order of our list. Every time we try to color a vertex it will have less than Δ colored neighbors so we can assign it a color. So we can color a graph with Δ colors if it isn't regular and isn't complete and isn't an odd cycle.
Case 2: Regular
If our graph is regular we use induction on the number of vertices. Let n be the number of vertices in our graph. When $n \leq \Delta$ we can color each vertex a different color. Let x and y be neighbors of a vertex v such that x and y aren't neighbors. Then let P be a path starting with vertices x, v, and y in that order. We can extend this path by adding a vertex v' to the path if all of it's neighbors are in the path already. We can extend this path until we cannot add any more vertices. Let's say we have all the vertices in our path at the end. Then we color x and y the same color. Now we pick a vertex which is adjacent to v' that isn't x or y. Let this vertex be v_i. We can color the vertices in the path starting at the vertex after y in order until v_i. Then we can color the remaining vertices in our path in reverse order. Finally we can color v'. If we P doesn't have all n vertices then let v_l be the last vertex in P. We know that all of the neighbors of v_l are in P. Let v_e be the neighbor of v_l that appears

the earliest in P. Then we know that all the vertices between v_e and v_l inclusive form a cycle C. Next, we can consider the subgraph $G' = G - C$. We can color G' with Δ colors by induction. Since our graph is connected we know that C and G' are connected. Let v_c be the vertex with latest index in our path such that it is connected to some u in G'. Then we know that $v_l \neq v_c$ because v_l has all its' neighbors in C. We know that v_{c+1} cannot be connected to u so we can assign it the same color as u. Then we can color the vertices v_{c+2} to v_l and then v_e to v_{c-1} and then finally color v_l. We can always color these vertices with Δ colors.

Therefore, every graph that isn't complete nor is an odd cycle is $\Delta-$colorable ■

Here is an example of a coloring of a regular graph with Δ colors.



Definition 25. An equitable coloring is an assignment of colors to vertices such that every pair of color sets differs in size by at least one.

4.8. **Unsolved Problems.** Several open problems remain in the field of graph coloring:

- **Equitable Coloring Conjecture:** Is it true that every graph with maximum degree $\Delta \geq 2$ has an equitable coloring using Δ colors, except for K_n with odd n and $\Delta + 1$ colors?
- **Total Coloring Conjecture:** Can every graph be colored using at most $\Delta + 2$ colors such that no adjacent or incident elements (vertices or edges) share a color?
- **Earth–Moon Problem:** What is the exact chromatic number of the union of two planar graphs?

4.9. **Real-world Applications.** Graph coloring has many practical uses:

- **Scheduling:** Assigning time slots to exams or tasks without conflicts. Equitable coloring ensures balanced workloads.
- **Frequency Assignment:** Allocating frequencies to transmitters to avoid interference. Equitable coloring balances frequency use.
- **Register Allocation:** Assigning variables to limited CPU registers in compilers, modeled by graph coloring.

5. Acknowledgements

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