An Introduction to Statistics, Moments, and the Central Limit Theorem

Pranav Krishnapuram

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Fundamental Definitions

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Sample: A subset of the population, taken to understand the population without having to know every element.

Statistic: A quantity computed from a sample, used to make inferences about population parameters.

Examples of Statistics

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Let $X_1, X_2, \ldots, X_n$ be a random sample from a population.

- **Sample Mean:** A measure of central tendency.

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

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- **Sample Mean:** A measure of central tendency.

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

- **Sample Standard Deviation:** A measure of dispersion or spread.

$$S = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2}$$

The Central Limit Theorem (CLT)

The Theorem

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with mean 0 and variance 1. Then, as $n \to \infty$, the distribution of the sum

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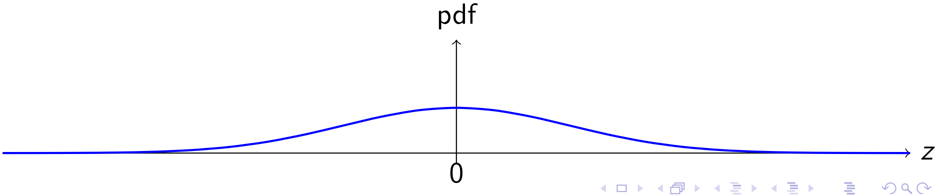

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- **1st Moment (Mean):** $\mu'_1 = E[X]$. This defines the center of the distribution.
- **2nd Moment:** $\mu'_2 = E[X^2]$. This is used to calculate the variance: $\text{Var}(X) = E[X^2] - (E[X])^2$.

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Generating Moments from the MGF

The moments of a distribution can be found by taking successive derivatives of the MGF at $t = 0$. The n^{th} moment about the origin, $E[X^n]$, is given by:

$$\frac{d^n}{dt^n} M_X(t) \bigg|_{t=0} = E[X^n]$$

Why does the MGF work?

The Taylor Series Connection

The property comes from the Taylor series expansion of e^{tX}:

$$e^{tX} = 1 + tX + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \dots$$

Taking the expectation of both sides gives:

$$M_X(t) = E[e^{tX}] = 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + \dots$$

This reveals the MGF as a power series in t, where the coefficient of $\frac{t^n}{n!}$ is precisely the n^{th} moment, $E[X^n]$.

Properties of MGFs

MGFs have some very useful properties that simplify calculations.

Sum of Independent Variables

If X and Y are independent random variables, then the MGF of their sum $Z = X + Y$ is the product of their individual MGFs:

$$M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX}e^{tY}] = E[e^{tX}]E[e^{tY}]$$

$$M_{X+Y}(t) = M_X(t)M_Y(t)$$

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Linear Transformation

For any constant c, the MGF of $Y = cX$ is:

$$M_{cX}(t) = E[e^{t(cX)}] = E[e^{(ct)X}]$$

$$M_{cX}(t) = M_X(ct)$$

Uniqueness of MGFs

The Uniqueness Property

A distribution is uniquely determined by its moment generating function. In other words, if two random variables have the same MGF, they must

have the same distribution. This property is fundamental to the proof of the Central Limit Theorem.

Example: Bernoulli and Binomial MGFs

Bernoulli Distribution

For $X \sim \text{Bernoulli}(p)$:

$$\begin{aligned} M_X(t) &= E[e^{tX}] \\ &= e^{t\cdot 1}(p) + e^{t\cdot 0}(1-p) \\ &= pe^t + 1 - p \end{aligned}$$

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Binomial Distribution

For $Y \sim \text{Binomial}(n, p)$, which is the sum of n i.i.d. Bernoulli trials:

$$\begin{aligned} M_Y(t) &= (M_X(t))^n \\ &= (pe^t + 1 - p)^n \end{aligned}$$

Proof of CLT using MGFs (Part 1)

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$$\begin{aligned} M_{Z_n}(t) &= M_{\frac{1}{\sqrt{n}} \sum X_i}(t) \\ &= M_{\sum X_i}\left(\frac{t}{\sqrt{n}}\right) \quad \text{(Linearity Property)} \\ &= \left[M_X\left(\frac{t}{\sqrt{n}}\right)\right]^n \quad \text{(Sum Property)} \end{aligned}$$

Proof of CLT using MGFs (Part 2)

Now, let's look at the Taylor expansion of $M_X(s)$ around $s = 0$:

$$M_X(s) = E[e^{sX}] = E\left[1 + sX + \frac{s^2X^2}{2!} + O(s^3)\right]$$

$$M_X(s) = 1 + sE[X] + \frac{s^2}{2}E[X^2] + O(s^3)$$

Since $E[X] = 0$ and $E[X^2] = \text{Var}(X) = 1$:

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Now substitute $s = \frac{t}{\sqrt{n}}$:

$$M_X\left(\frac{t}{\sqrt{n}}\right) = 1 + \frac{1}{2}\left(\frac{t}{\sqrt{n}}\right)^2 + O\left(\frac{t^3}{n^{3/2}}\right) = 1 + \frac{t^2}{2n} + O(n^{-3/2})$$

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Finally, we take the limit of the full MGF:

$$\lim_{n\to\infty} M_{Z_n}(t) = \lim_{n\to\infty} \left[1 + \frac{t^2}{2n} + O(n^{-3/2})\right]^n$$

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This is the MGF of a standard normal distribution. By the uniqueness property of MGFs, Z_n converges in distribution to $N(0,1)$. ■

Thank you!