Spherical Harmonics as Eigenfunctions of the Laplace-Beltrami Operator

Palaash Gang

Euler Circle, Mountain View, CA 94040

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Motivation and Setup

The *Laplacian* is a second-order differential operator

- Measures how a function differs from its average value nearby
- In (flat)3D space: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

On curved surfaces we must account for the geometry of space and hence use the Laplace-Beltrami operator.

Laplace-Beltrami Operator

On the 2-sphere $\mathbb{S}^2$ it is defined as

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

It's eigenfunctions are spherical harmonics, and these form a complete basis for square-integrable functions ($\int_D |f(x)|^2\, dx < \infty$).

We seek all functions $Y(\theta, \phi)$ satisfying: $\Delta_{\mathbb{S}^2} Y = -\lambda Y$

Formal Definition

Formal definition of Spherical Harmonics

Spherical harmonics $Y_\ell^m : \mathbb{S}^2 \to \mathbb{C}$ are defined by

$$Y_\ell^m(\theta, \phi) = N_\ell^m \, P_\ell^m(\cos\theta) \, e^{im\phi}, \quad N_\ell^m = \sqrt{\frac{2\ell+1}{4\pi} \cdot \frac{(\ell - |m|)!}{(\ell + |m|)!}}$$

- $\theta \in [0, \pi]$ is the polar angle
- and $\phi \in [0, 2\pi)$ is the azimuthal angle
- $\ell \in \mathbb{Z}_{\geq 0}$, and $m \in \{-\ell, -\ell+1, \ldots, \ell\}$
- $P_\ell^m(x)$ is the associated Legendre function
- N_ℓ^m is the normalization constant

Orthonormality Theorem

Theorem (Orthonormality of Spherical Harmonics)

The spherical harmonics $Y_\ell^m(\theta, \phi)$ are orthonormal in $L^2(\mathbb{S}^2)$, i.e.,

$$\int_{\mathbb{S}^2} Y_\ell^m(\theta, \phi)\, \overline{Y_{\ell'}^{m'}(\theta, \phi)}\, d\Omega = \delta_{\ell\ell'}\delta_{mm'}$$

where $d\Omega = \sin\theta \, d\theta \, d\phi$.

- This means they form an orthonormal basis on the space of square-integrable functions over the sphere.
- Proven using properties of associated Legendre polynomials and orthogonality of complex exponentials.

Proof- (1)

We evaluate the inner product:

$$\langle Y_\ell^m, Y_{\ell'}^{m'} \rangle = \int_0^{2\pi} \int_0^{\pi} Y_\ell^m(\theta, \phi)\, \overline{Y_{\ell'}^{m'}(\theta, \phi)}\, \sin\theta \, d\theta \, d\phi$$

Substitute definitions:

$$= N_\ell^m N_{\ell'}^{m'} \int_0^{2\pi} e^{i(m-m')\phi} \, d\phi \cdot \int_0^{\pi} P_\ell^m(\cos\theta) P_{\ell'}^{m'}(\cos\theta) \sin\theta \, d\theta$$

The first integral is:

$$\int_0^{2\pi} e^{i(m-m')\phi} \, d\phi = 2\pi \delta_{mm'}$$

Now use the change of variable $x = \cos\theta \Rightarrow dx = -\sin\theta \, d\theta$ in the second integral.

Proof- (2)

$$\int_0^\pi P_\ell^m(\cos\theta) P_{\ell'}^m(\cos\theta)\sin\theta\, d\theta = \int_{-1}^1 P_\ell^m(x) P_{\ell'}^m(x)\, dx$$

Using the orthogonality of associated Legendre functions:

$$\int_{-1}^1 P_\ell^m(x) P_{\ell'}^m(x)\, dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'}$$

Putting everything together:

$$\langle Y_\ell^m, Y_{\ell'}^{m'} \rangle = 2\pi N_\ell^m N_{\ell'}^{m'} \cdot \frac{2}{2\ell+1} \cdot \frac{(\ell+m)!}{(\ell-m)!} \cdot \delta_{mm'}\delta_{\ell\ell'}$$

This evaluates to 1 by the definition of the normalization constant.

Completeness Theorem

Theorem (Completeness of Spherical Harmonics)

Any square-integrable function $f \in L^2(\mathbb{S}^2)$ can be expanded as

$$f(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_\ell^m Y_\ell^m(\theta, \phi)$$

with convergence in the L^2-norm.

- This shows that the $\{Y_\ell^m\}$ form a complete orthonormal basis.
- Coefficients are computed by projection:

$$a_\ell^m = \int_{\mathbb{S}^2} f(\theta, \phi) \, \overline{Y_\ell^m(\theta, \phi)} \, d\Omega$$

Expansion Theorem

Theorem (Spherical Harmonic Expansion)

If $f \in L^2(\mathbb{S}^2)$*, then*

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_\ell^m Y_\ell^m(\theta,\phi), \quad \textit{where} \quad a_\ell^m = \langle f, Y_\ell^m \rangle$$

and

$$\|f\|^2 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} |a_\ell^m|^2 \quad \textit{(Parseval's Identity).}$$

- Spherical harmonics act as Fourier modes on the sphere.
- This expansion is used in physics, signal processing, and solving PDEs on the sphere.

Hyperspherical Harmonics

- On $\mathbb{S}^n$, we define harmonics as eigenfunctions of the Laplace–Beltrami operator on the n-sphere:

$$\Delta_{\mathbb{S}^n} Y = -\lambda Y$$

- Eigenvalues: $\lambda = \ell(\ell + n - 1)$
- Dimension of eigenspace:

$$\dim(\mathcal{H}_\ell(\mathbb{S}^n)) = \binom{n+\ell}{\ell} - \binom{n+\ell-2}{\ell-2}$$

- These generalize the classical Y_ℓ^m and appear in quantum mechanics and higher-dimensional PDEs.

Generalized Expansion

- Any function $f \in L^2(\mathbb{S}^n)$ can be written as:

$$f = \sum_{\ell=0}^{\infty} \sum_{j=1}^{d_\ell} a_\ell^j Y_\ell^j, \quad a_\ell^j = \langle f, Y_\ell^j \rangle$$

  where Y_ℓ^j runs over an orthonormal basis for $\mathcal{H}_\ell(\mathbb{S}^n)$.
- Parseval's still holds:

$$\|f\|^2 = \sum_{\ell=0}^{\infty} \sum_{j=1}^{d_\ell} |a_\ell^j|^2$$

- These expansions are used in machine learning on manifolds, cosmology, and mathematical physics.

Conclusion

- Spherical harmonics arise as eigenfunctions of the Laplace–Beltrami operator on $\mathbb{S}^2$
- They form a complete orthonormal basis for $L^2(\mathbb{S}^2)$
- Enable spectral decompositions and solve PDEs on the sphere
- Generalize beautifully to higher-dimensional spheres $\mathbb{S}^n$
- Underlie many applications across physics, geometry, and signal processing

Thank you!, Questions?