SPHERICAL HARMONICS AS EIGENFUNCTIONS OF THE LAPLACE-BELTRAMI OPERATOR

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Abstract. This paper constructs spherical harmonics as eigenfunctions of the Laplace-Beltrami operator on the sphere. Starting from first principles, we derive their explicit form and establish orthonormality and completeness in $L^2(\mathbb{S}^2)$. We then prove key structural results including the Addition Theorem and reproducing kernel formulation, and conclude by generalizing to higher-dimensional spheres. Finally, we extend the framework to higher-dimensional spheres by constructing hyperspherical harmonics and generalizing the expansion theory to an n-sphere.

1. Introduction

Spherical harmonics arise from the question: how can we describe functions defined on the surface of a sphere? They thus provide a powerful framework for mathematical physics. These functions, arising naturally as solutions to Laplace's equation in spherical coordinates, generalise the Fourier series to spherical geometries. Their significance lies in their ability to decompose functions on the sphere into orthogonal components, proving useful across diverse fields such as physics, geophysics, computer graphics, and quantum mechanics.

Historically we can trace their study to the 18th century, with early contributions from Pierre-Simon Laplace and Adrien-Marie Legendre. Legendre's work on polynomial solutions laid the groundwork for what we now call Legendre polynomials, which form the basis of spherical harmonics. By the 19th century, these functions were formalized by scientists like William Thomson (Lord Kelvin) and James Clerk Maxwell, who applied them to problems in electromagnetism and gravitational potential theory.

Bringing us to the main question, why care about spherical harmonics? They're a natural fit for problems with rotational symmetry or angular behavior in 3D space. Formally, they form a complete orthonormal basis for the space of square-integrable functions on the unit sphere, $L^2(S^2)$, meaning any function $f(\theta, \phi)$ on the sphere can be expressed as $f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} Y_l^m(\theta, \phi)$, where Y_l^m are the spherical harmonics and c_{lm} are coefficients. This property makes them a powerhouse for spectral analysis, especially in today's world of big data and computational modeling, from studying Earth's magnetic field to rendering realistic lighting in video games.

This paper provides a comprehensive and introductory exposition of spherical harmonics, beginning with their derivation from the Laplace-Beltrami operator on S^2 and their fundamental properties, including orthonormality, completeness, and the expansion of functions. We delve into key results such as Parseval's identity, the addition theorem, and the role of

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zonal harmonics and reproducing kernels. The discussion then extends to spherical harmonics on S^n, covering their definition, orthonormality, completeness, and expansion properties in higher dimensions. This paper aims to present these results in a clear and rigorous framework.

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2. Preliminaries: Mathematical Framework

We work in the Euclidean space $\mathbb{R}^n$, the set of all ordered n-tuples of real numbers: $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$, equipped with the standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i$, and norm $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$. Within $\mathbb{R}^3$, the *unit 2-sphere* is defined as

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

For functions defined on $\mathbb{S}^2$, it is convenient to use *spherical coordinates*, related to Cartesian coordinates by: $x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta$, where $r \geq 0$, $\theta \in [0, \pi]$ is the polar angle, and $\phi \in [0, 2\pi)$ is the azimuthal angle. Restricting to the unit sphere $r = 1$, the coordinate transformation $(\theta, \phi) \mapsto (x, y, z)$ has Jacobian determinant equal to $\sin\theta$, which gives rise to the *surface element* $d\Omega = \sin\theta \, d\theta \, d\phi$.

We work in the space $L^2(\mathbb{S}^2)$, consisting of complex-valued measurable functions

$$f : \mathbb{S}^2 \to \mathbb{C} \quad \text{for which} \int_{\mathbb{S}^2} |f(\theta, \phi)|^2 \, d\Omega < \infty.$$

The space $L^2(\mathbb{S}^2)$ consists of all complex-valued measurable functions defined on the unit sphere $\mathbb{S}^2$ such that the square of their absolute value is integrable, and finite, over the sphere. The spherical harmonics $Y_\ell^m(\theta, \phi)$ form a complete orthonormal basis for $L^2(\mathbb{S}^2)$.

$$L^2(\mathbb{S}^2) = \left\{ f : \mathbb{S}^2 \to \mathbb{C} \;\middle|\; \int_{\mathbb{S}^2} |f(\theta, \phi)|^2 \, d\Omega < \infty \right\}.$$

The inner product on this space is

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) \overline{g(\theta, \phi)} \sin\theta \, d\theta \, d\phi,$$

with corresponding norm $\|f\| = \sqrt{\langle f, f \rangle}$. Two functions f, g are *orthogonal* if $\langle f, g \rangle = 0$, and a set $\{\phi_n\}$ is *orthonormal* if

$$\langle \phi_n, \phi_m \rangle = \delta_{nm}, \quad \text{where} \quad \delta_{nm} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

The *Laplacian* in $\mathbb{R}^3$ is defined by

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{2.1}$$

In spherical coordinates, this becomes

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{2.2}$$

We consider the *eigenvalue problem*

$$\Delta_{\mathbb{S}^2} f + \lambda f = 0, \quad \text{where} \quad \lambda \in \mathbb{R}$$

is the *eigenvalue* and $f \not\equiv 0$ is the corresponding *eigenfunction*. The collection of such eigenfunctions forms a complete orthonormal basis for $L^2(\mathbb{S}^2)$, and the eigenvalues are nonpositive reals. To solve this problem, we use *separation of variables*, assuming a product solution $f(\theta, \phi) = \Theta(\theta)\Phi(\phi)$. Substituting into the eigenvalue equation separates it into two ODEs. The azimuthal part yields:

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0,$$

whose general solution is $\Phi(\phi) = e^{im\phi} = \text{cis}(m\phi)$, $m \in \mathbb{Z}$. The polar equation becomes:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + \lambda\Theta = 0.$$

By substituting $x = \cos\theta$, this transforms into the *associated Legendre differential equation*:

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0. \tag{2.3}$$

Solutions to this equation are the *associated Legendre functions* $P_\ell^m(x)$, defined for integers $\ell \geq 0$, $|m| \leq \ell$, by

$$P_\ell^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_\ell(x),$$

where $P_\ell(x)$ are the *Legendre polynomials* [11](c.4), satisfying

$$(1-x^2)\frac{d^2P_\ell}{dx^2} - 2x\frac{dP_\ell}{dx} + \ell(\ell+1)P_\ell(x) = 0.$$

The Legendre polynomials are orthogonal on the interval $[-1, 1]$ with respect to the constant weight $p(x) = 1$:

$$\int_{-1}^{1} P_\ell(x)P_k(x)\,dx = 0 \quad \text{if } \ell \neq k.$$

The functions $P_\ell^m(\cos\theta)$ and $e^{im\phi}$ combine to form the basis of the *spherical harmonics*, which solve the Laplace–Beltrami eigenproblem and span $L^2(\mathbb{S}^2)$. Their construction and properties will be the subject of the next sections.

3. Deriving the Spherical Harmonics from the Laplace–Beltrami Operator

3.1. **Motivation and Setup.** The Laplacian is a differential operator that quantifies how much a function differs from the average of nearby values. It is defined as the divergence of the gradient: $\Delta f = \nabla \cdot (\nabla f)$ and invariant under rotations and translations. Because it detects local imbalances, it plays a central role to various branches of physics. However, on curved surfaces, the standard Laplacian no longer captures these properties correctly, as it fails to account for the geometry of the surface. Hence the Laplace-Beltrami operator is used to make the gradient and divergence intrinsic to the geometry of the surface.

3.2. **Laplace-Beltrami Operator.** In $\mathbb{R}^3$, Laplace's equation is given by $\nabla^2 f = 0$. For problems exhibiting spherical symmetry, we express the Laplacian in spherical coordinates. By applying the chain rule and expressing partial derivatives with respect to x, y, z in terms of r, θ, ϕ, one obtains the standard form (2.2):

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}.$$

A detailed derivation is computational and standard, and can be found in many math references. [13]. On the unit sphere $r = 1$, the radial part drops out, and we define the *Laplace–Beltrami operator* on $\mathbb{S}^2$ as

$$\Delta_{\mathbb{S}^2} f = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}. \tag{3.1}$$

3.3. **Eigenvalue problem, separation of variables.** We now study the eigenvalue problem for the *Laplace-Beltrami operator* on the sphere. Given a smooth function $f : \mathbb{S}^2 \to \mathbb{C}$, we wish to find all solutions to the partial differential equation $\Delta_{\mathbb{S}^2} f + \lambda f = 0$, where $\lambda \in \mathbb{R}$ is an eigenvalue. To solve this analytically, we assume that the eigenfunctions admit a separable form:

$$f(\theta, \phi) = \Theta(\theta)\Phi(\phi),$$

where Θ depends only on the polar angle θ, and Φ on the azimuthal angle ϕ. Substituting into the operator yields:

$$\Delta_{\mathbb{S}^2} f = \left(\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right)\Phi(\phi) + \left(\frac{1}{\sin^2\theta}\frac{d^2\Phi}{d\phi^2}\right)\Theta(\theta).$$

Substituting into the eigenvalue equation and dividing by $\Theta(\theta)\Phi(\phi)$, we obtain:

$$\frac{1}{\Theta}\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right) + \frac{1}{\Phi}\left(\frac{1}{\sin^2\theta}\frac{d^2\Phi}{d\phi^2}\right) + \lambda = 0.$$

This expression separates variables: the first term depends only on θ, and the second only on ϕ. As a result, each term must equal a constant. We write $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2$, for some separation constant $m \in \mathbb{R}$, and thus obtain the two decoupled equations:

- Azimuthal equation:

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0, \tag{3.2}$$

- Polar equation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + \lambda\Theta = 0. \tag{3.3}$$

To ensure that the solution $f(\theta, \phi)$ is single-valued on the sphere, we impose periodicity in ϕ: $\Phi(\phi + 2\pi) = \Phi(\phi)$, which implies that $m \in \mathbb{Z}$. We will analyze each of these ODEs in the following subsections.

3.4. **Solving the Azimuthal Equation.** From separation of variables, the azimuthal component $\Phi(\phi)$ satisfies (3.2)

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0.$$

The general solution is $\Phi(\phi) = Ce^{im\phi} + De^{-im\phi}, \quad m \in \mathbb{R}$.To ensure $f(\theta, \phi) = \Theta(\theta)\Phi(\phi)$ is single-valued and continuous on the sphere, we impose $\Phi(\phi+2\pi) = \Phi(\phi)$, which holds if and only if $m \in \mathbb{Z}$.

We define the normalized azimuthal basis functions

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}, \quad m \in \mathbb{Z}, \quad \text{which satisfy} \quad \int_0^{2\pi} \Phi_m(\phi)\overline{\Phi_n(\phi)}\, d\phi = \delta_{mn}.$$

These will serve as the azimuthal factors in the spherical harmonics.

3.5. **Solving the Polar Equation: Associated Legendre Functions.** After separating the azimuthal variable, the remaining ODE for the polar component $\Theta(\theta)$ becomes (3.3):

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + \lambda\Theta = 0.$$

This is a second-order linear ODE with singularities at $\theta = 0$ and $\theta = \pi$, corresponding (unsurprisingly!) to the poles of the sphere. To simplify the analysis, we perform a change in variables by letting $x = \cos\theta$, so that $x \in [-1, 1]$. Using the chain rule:

$$\frac{d\Theta}{d\theta} = -\sin\theta\frac{d\Theta}{dx}, \quad \frac{d^2\Theta}{d\theta^2} = -\cos\theta\frac{d\Theta}{dx} + \sin^2\theta\frac{d^2\Theta}{dx^2}.$$

Substituting into the equation transforms it into:

$$(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \left[\lambda - \frac{m^2}{1-x^2}\right]\Theta = 0.$$

This is the *associated Legendre differential equation*, and we denote its solutions by $P_\ell^m(x)$, where $\ell \geq |m|$ is an integer and $\lambda = \ell(\ell+1)$. These conditions arise from requiring:

- Finite behavior at the endpoints $x = \pm 1$,
- Polynomial solutions to ensure regularity at the poles,
- Orthogonality under a suitable weight function $p(x) = 1$:

$$\int_{-1}^{1} P_\ell^m(x)\, P_k^m(x)\, dx = 0 \quad \text{if } \ell \neq k.$$

The general solution is (Rodrigues' formula):

$$P_\ell^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_\ell(x),$$

where $P_\ell(x)$ is the Legendre polynomial of degree ℓ, satisfying:

$$(1-x^2)\frac{d^2P_\ell}{dx^2} - 2x\frac{dP_\ell}{dx} + \ell(\ell+1)P_\ell(x) = 0.$$

By solving the polar equation, we obtain the complete set of angular functions needed for constructing spherical harmonics, which take the form

$$Y_\ell^m(\theta, \phi) = N_\ell^m P_\ell^m(\cos\theta)e^{im\phi}, \tag{3.4}$$

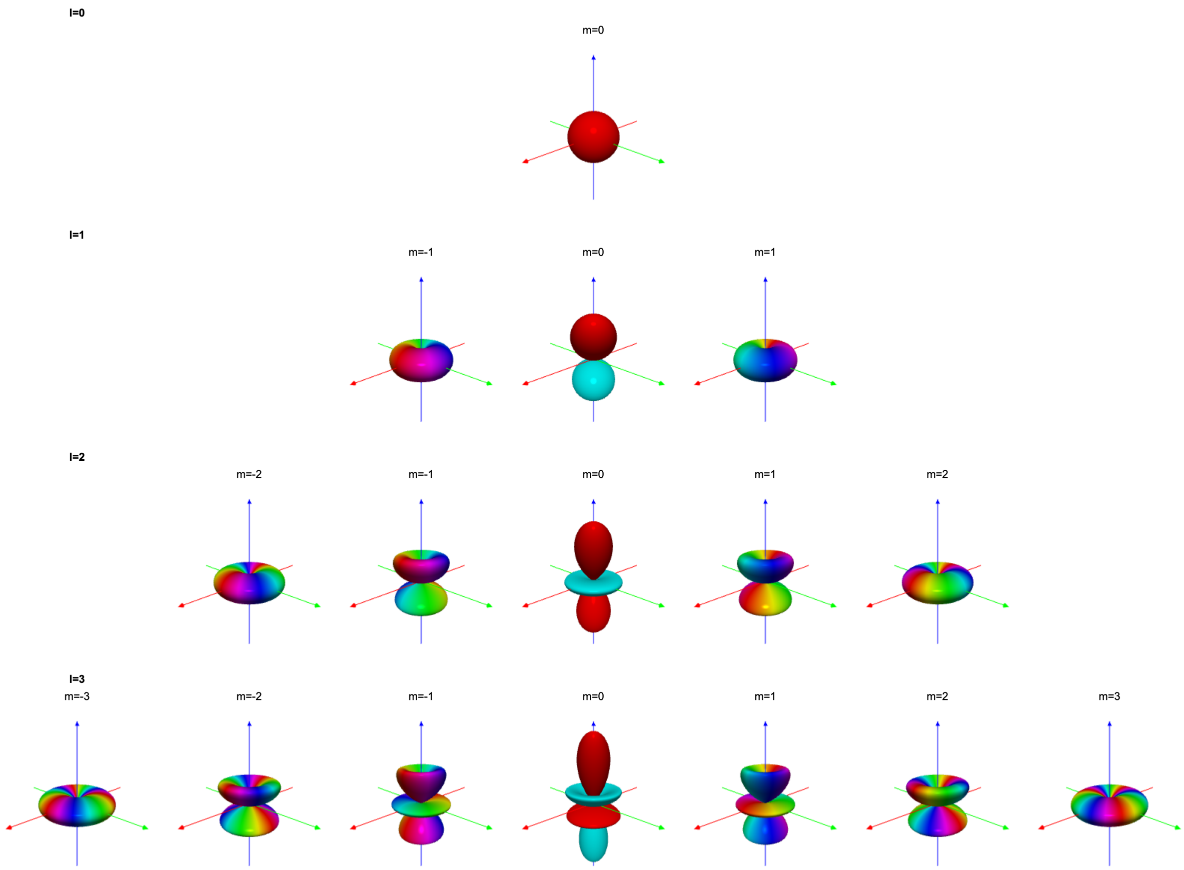

Figure 1. Visualisations of $Y_{\ell m}^{\text{real}}$

where N_ℓ^m is a normalization constant. More specifically,

$$N_\ell^m = \sqrt{\frac{2\ell+1}{4\pi} \cdot \frac{(\ell-m)!}{(\ell+m)!}} \tag{3.5}$$

We will derive this result in subsequent sections.

3.6. **Visualisations [14].** To develop an intuitive understanding of spherical harmonics, Fig 1 presents the first few real-valued spherical harmonic functions up to degree $\ell = 3$. The real-valued spherical harmonics are defined as:

$$Y_{\ell m}^{\text{real}}(\theta, \phi) = \begin{cases} \sqrt{2}\, N_{\ell m}\, P_\ell^m(\cos\theta)\cos(m\phi), & m > 0, \\ N_{\ell 0}\, P_\ell^0(\cos\theta), & m = 0, \\ \sqrt{2}\, N_{\ell |m|}\, P_\ell^{|m|}(\cos\theta)\sin(|m|\phi), & m < 0, \end{cases}$$

where P_ℓ^m denotes the associated Legendre function and $N_{\ell m}$ is a normalization constant that ensures orthonormality on the sphere.

ℓ	m	$Y_{\ell m}^{\text{real}}(\theta, \phi)$
0	0	$\sqrt{\frac{1}{4\pi}}$
1	-1	$\sqrt{\frac{3}{4\pi}}\sin\theta\sin\phi$
	0	$\sqrt{\frac{3}{4\pi}}\cos\theta$
	1	$\sqrt{\frac{3}{4\pi}}\sin\theta\cos\phi$
2	-2	$\sqrt{\frac{15}{4\pi}}\sin^2\theta\sin(2\phi)$
	-1	$\sqrt{\frac{15}{4\pi}}\sin\theta\cos\theta\sin\phi$
	0	$\sqrt{\frac{5}{16\pi}}(3\mathbb{S}^2\theta - 1)$
	1	$\sqrt{\frac{15}{4\pi}}\sin\theta\cos\theta\cos\phi$
	2	$\sqrt{\frac{15}{4\pi}}\sin^2\theta\cos(2\phi)$
3	-3	$\sqrt{\frac{35}{2\pi}}\sin^3\theta\sin(3\phi)$
	-2	$\sqrt{\frac{105}{4\pi}}\sin^2\theta\cos\theta\sin(2\phi)$
	-1	$\sqrt{\frac{21}{64\pi}}\sin\theta(5\mathbb{S}^2\theta - 1)\sin\phi$
	0	$\sqrt{\frac{7}{16\pi}}\cos\theta(5\mathbb{S}^2\theta - 3)$
	1	$\sqrt{\frac{21}{64\pi}}\sin\theta(5\mathbb{S}^2\theta - 1)\cos\phi$
	2	$\sqrt{\frac{105}{4\pi}}\sin^2\theta\cos\theta\cos(2\phi)$
	3	$\sqrt{\frac{35}{2\pi}}\sin^3\theta\cos(3\phi)$

Table 1. Real spherical harmonics $Y_{\ell m}^{\text{real}}(\theta, \phi)$ for degrees $\ell = 0$ to 3

To visualize these functions in Fig 1 , we can represent the value of the function as the radius. Each spherical harmonic's nodal structure and symmetry become apparent in these plots. For example, functions with $|m| > 0$ display azimuthal variation (variation in longitude-azimuthal angle ϕ), while $m = 0$ modes are zonal (variation in latitude- polar angle θ) and symmetric around the z-axis.

4. Orthonormality, Completeness and Basic Properties

4.1. **Formal definition of Spherical Harmonics.** Let $\ell \in \mathbb{Z}_{\geq 0}$, and $m \in \{-\ell, -\ell + 1, \ldots, \ell\}$. Spherical harmonics $Y_\ell^m : \mathbb{S}^2 \to \mathbb{C}$ are defined by (3.4):

$$Y_\ell^m(\theta, \phi) = N_\ell^m \, P_\ell^m(\cos\theta) \, e^{im\phi},$$

where:

- $\theta \in [0, \pi]$ is the polar angle,
- $\phi \in [0, 2\pi)$ is the azimuthal angle,
- $P_\ell^m(x)$ is the associated Legendre function of degree ℓ and order m,

- N_ℓ^m is the normalization constant given by (3.5)

$$N_\ell^m = \sqrt{\frac{2\ell+1}{4\pi} \cdot \frac{(\ell-|m|)!}{(\ell+|m|)!}}.$$

4.2. **Orthonormality and Derivation of the Normalization Constant.**

Theorem 4.1. *[Orthonormality of Spherical Harmonics] The spherical harmonics* $Y_\ell^m(\theta,\phi) = N_\ell^m\, P_\ell^m(\cos\theta)\, e^{im\phi}$ *are orthonormal in* $L^2(\mathbb{S}^2)$ *with respect to the inner product*

$$\langle f, g\rangle = \int_0^{2\pi} \int_0^{\pi} f(\theta,\phi)\, \overline{g(\theta,\phi)}\, \sin\theta \, d\theta \, d\phi,$$

provided that the normalization constant N_ℓ^m *is chosen as* $N_\ell^m = \sqrt{\frac{2\ell+1}{4\pi} \cdot \frac{(\ell-m)!}{(\ell+m)!}}$.

Proof. We compute the inner product of two spherical harmonics:

$$\langle Y_\ell^m, Y_k^n\rangle = \int_0^{2\pi} \int_0^{\pi} Y_\ell^m(\theta,\phi)\, \overline{Y_k^n(\theta,\phi)}\, \sin\theta \, d\theta \, d\phi.$$

Substituting the definitions:

$$= N_\ell^m \overline{N_k^n} \int_0^{2\pi} e^{i(m-n)\phi}\, d\phi \cdot \int_0^{\pi} P_\ell^m(\cos\theta) P_k^n(\cos\theta)\, \sin\theta \, d\theta.$$

Make the substitution $x = \cos\theta$, $dx = -\sin\theta \, d\theta$, so the second integral becomes:

$$\int_{-1}^{1} P_\ell^m(x) P_k^n(x)\, dx.$$

Now apply orthogonality of exponentials and associated Legendre functions:

$$\int_0^{2\pi} e^{i(m-n)\phi}\, d\phi = 2\pi \delta_{mn},$$

$$\int_{-1}^{1} P_\ell^m(x) P_k^m(x)\, dx = \frac{2(\ell+m)!}{(2\ell+1)(\ell-m)!}\delta_{\ell k}.$$

Combining both:

$$\langle Y_\ell^m, Y_k^n\rangle = |N_\ell^m|^2 \cdot 2\pi \cdot \frac{2(\ell+m)!}{(2\ell+1)(\ell-m)!}\delta_{\ell k}\delta_{mn}.$$

To ensure orthonormality, we require: $\langle Y_\ell^m, Y_k^n\rangle = \delta_{\ell k}\delta_{mn}$.
Thus, we solve for the modulus squared of the normalization constant:

$$|N_\ell^m|^2 = \frac{(2\ell+1)}{4\pi} \cdot \frac{(\ell-m)!}{(\ell+m)!},$$

and take the positive square root:

$$N_\ell^m = \sqrt{\frac{2\ell+1}{4\pi} \cdot \frac{(\ell-m)!}{(\ell+m)!}}.$$

Remark 4.2. Once the correct normalization is known, orthonormality may alternatively be verified directly by substituting into the integral form, the statement and proof of which follows in 4.3

Theorem 4.3. *The complex spherical harmonics* $\{Y_\ell^m\}_{\ell\geq 0,\ -\ell\leq m\leq \ell}$ *satisfy the orthonormality relation*

$$\int_0^{2\pi}\int_0^{\pi} Y_\ell^m(\theta,\phi)\,\overline{Y_{\ell'}^{m'}(\theta,\phi)}\,\sin\theta\,d\theta\,d\phi = \delta_{\ell\ell'}\,\delta_{mm'} \tag{4.1}$$

Proof. We compute:

$$\langle Y_\ell^m, Y_{\ell'}^{m'}\rangle = \int_0^{2\pi}\int_0^{\pi} Y_\ell^m(\theta,\phi)\,\overline{Y_{\ell'}^{m'}(\theta,\phi)}\,\sin\theta\,d\theta\,d\phi.$$

Substituting the definitions, $Y_\ell^m(\theta,\phi) = N_\ell^m P_\ell^{|m|}(\cos\theta)e^{im\phi}, \quad \overline{Y_{\ell'}^{m'}(\theta,\phi)} = N_{\ell'}^{m'} P_{\ell'}^{|m'|}(\cos\theta)e^{-im'\phi}$, so the product becomes:

$$\langle Y_\ell^m, Y_{\ell'}^{m'}\rangle = N_\ell^m N_{\ell'}^{m'} \int_0^{2\pi} e^{i(m-m')\phi}\,d\phi \int_0^{\pi} P_\ell^{|m|}(\cos\theta)P_{\ell'}^{|m'|}(\cos\theta)\sin\theta\,d\theta.$$

From standard orthogonality relations:

$$\int_0^{2\pi} e^{i(m-m')\phi}\,d\phi = 2\pi\,\delta_{mm'} \quad \text{and} \quad \int_0^{\pi} P_\ell^{|m|}(\cos\theta)P_{\ell'}^{|m|}(\cos\theta)\sin\theta\,d\theta = \frac{2}{2\ell+1}\cdot\frac{(\ell+|m|)!}{(\ell-|m|)!}\,\delta_{\ell\ell'}.$$

Thus,

$$\langle Y_\ell^m, Y_{\ell'}^{m'}\rangle = N_\ell^m N_{\ell'}^{m'}\cdot 2\pi\cdot\delta_{mm'}\cdot\frac{2}{2\ell+1}\cdot\frac{(\ell+|m|)!}{(\ell-|m|)!}\cdot\delta_{\ell\ell'}.$$

When $\ell=\ell'$ and $m=m'$, we substitute $(N_\ell^m)^2$:

$$(N_\ell^m)^2 = \frac{2\ell+1}{4\pi}\cdot\frac{(\ell-|m|)!}{(\ell+|m|)!},$$

so

$$\langle Y_\ell^m, Y_\ell^m\rangle = \left(\frac{2\ell+1}{4\pi}\cdot\frac{(\ell-|m|)!}{(\ell+|m|)!}\right)\cdot 2\pi\cdot\frac{2}{2\ell+1}\cdot\frac{(\ell+|m|)!}{(\ell-|m|)!} = 1.$$

Therefore, $\langle Y_\ell^m, Y_{\ell'}^{m'}\rangle = \delta_{\ell\ell'}\delta_{mm'}$.

4.3. Completeness of Y_ℓ^m.

Further, we state that the spherical harmonics $\{Y_\ell^m(\theta,\phi)\}$, for $\ell\geq 0$ and $-\ell\leq m\leq\ell$, form a complete orthonormal system in $L^2(\mathbb{S}^2)$.

Theorem 4.4. *[Completeness theorem] The set of spherical harmonics* $\{Y_\ell^m : \ell\geq 0, -\ell\leq m\leq\ell\}$ *forms a complete orthonormal system in* $L^2(\mathbb{S}^2)$.

Lemma 4.5. *Let* $f\in L^2(\mathbb{S}^2)$*, and suppose that*

$$\langle f, Y_\ell^m\rangle = \int_{\mathbb{S}^2} f(\theta,\phi)\,\overline{Y_\ell^m(\theta,\phi)}\,d\Omega = 0, \quad \{\forall\ell\geq 0, -\ell\leq m\leq\ell\}.$$

Then $f=0$ *in* $L^2(\mathbb{S}^2)$.

Proof. Let $V\subset L^2(\mathbb{S}^2)$ be the subspace of finite linear combinations of spherical harmonics: $V=\operatorname{span}\{Y_\ell^m\}$. If $\langle f, Y_\ell^m\rangle = 0$ for all ℓ, m, then $f\in V^\perp$, the orthogonal complement of V. Since the spherical harmonics are eigenfunctions of the Laplace-Beltrami operator and form a countable orthonormal set, the set V is dense in $L^2(\mathbb{S}^2)$. Therefore, the only function orthogonal to all of V is the zero function, and hence $f=0$ in L^2.

Proof of Theorem 4.4. We have already shown that the spherical harmonics form an orthonormal set in $L^2(\mathbb{S}^2)$. By the lemma above, the only function orthogonal to all of them is the zero function. This means the orthogonal complement of their span is trivial, so their span is dense in $L^2(\mathbb{S}^2)$. Hence, the spherical harmonics form a complete orthonormal system.

4.4. **Basic Properties.**

Claim 4.6. *The spherical harmonics Y_ℓ^m satisfy $\Delta_{\mathbb{S}^2} Y_\ell^m = -\ell(\ell+1)Y_\ell^m$.*

Proof. We write the spherical harmonic (3.4) as $Y_\ell^m(\theta,\phi) = P_\ell^m(\cos\theta)\cdot e^{im\phi}$. Applying the Laplace-Beltrami operator (3.1) gives:

$$\Delta_{\mathbb{S}^2} Y_\ell^m = \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}P_\ell^m(\cos\theta)\right) + \frac{1}{\sin^2\theta}\cdot\frac{\partial^2}{\partial\phi^2}\right]\cdot e^{im\phi}.$$

Since $\frac{\partial^2}{\partial\phi^2}e^{im\phi} = -m^2 e^{im\phi}$, the second term becomes:

$$-\frac{m^2}{\sin^2\theta}P_\ell^m(\cos\theta)\cdot e^{im\phi}.$$

For the first term, we use the identity for associated Legendre functions (2.3):

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}P_\ell^m(\cos\theta)\right) = -\ell(\ell+1)P_\ell^m(\cos\theta) + \frac{m^2}{\sin^2\theta}P_\ell^m(\cos\theta).$$

Multiplying by $e^{im\phi}$, the first term becomes:

$$\left(-\ell(\ell+1)P_\ell^m + \frac{m^2}{\sin^2\theta}P_\ell^m\right)e^{im\phi}.$$

Adding both parts, we obtain:

$$\Delta_{\mathbb{S}^2} Y_\ell^m = \left[-\ell(\ell+1)P_\ell^m + \frac{m^2}{\sin^2\theta}P_\ell^m\right]e^{im\phi} - \frac{m^2}{\sin^2\theta}P_\ell^m e^{im\phi}.$$

The $\frac{m^2}{\sin^2\theta}$ terms cancel, yielding: $\Delta_{\mathbb{S}^2} Y_\ell^m = -\ell(\ell+1)P_\ell^m(\cos\theta)e^{im\phi} = -\ell(\ell+1)Y_\ell^m(\theta,\phi)$.

Remark 4.7. Note that Y_ℓ^m is an eigenfunction to the laplace-beltrami operator $\Delta_{\mathbb{S}^2}$ with the corresponding eigenvalue $\ell(\ell+1)$.

Claim 4.8. *The spherical harmonics satisfy $\overline{Y_\ell^m(\theta,\phi)} = (-1)^m Y_\ell^{-m}(\theta,\phi)$.*

Proof. Using the definition $Y_\ell^m(\theta,\phi) = N_\ell^m P_\ell^m(\cos\theta)e^{im\phi}$, we compute
the complex conjugate: $\overline{Y_\ell^m(\theta,\phi)} = N_\ell^m P_\ell^m(\cos\theta)e^{-im\phi}$.
From the identity $N_\ell^{-m} = (-1)^m N_\ell^m$, it follows that $\overline{Y_\ell^m(\theta,\phi)} = (-1)^m Y_\ell^{-m}(\theta,\phi)$.

Theorem 4.9. *For all integers $\ell \geq 0$ and $-\ell \leq m \leq \ell$, the spherical harmonic $Y_\ell^m(\theta,\phi)$ satisfies the identity:*

$$Y_\ell^m(\pi-\theta,\phi+\pi) = (-1)^\ell Y_\ell^m(\theta,\phi). \tag{4.2}$$

Proof. We use the explicit expression for spherical harmonics (3.4): $Y_\ell^m(\theta,\phi) = N_{\ell m}P_\ell^m(\cos\theta)\,e^{im\phi}$, Under the transformation $(\theta,\phi)\mapsto(\pi-\theta,\phi+\pi)$, we have:

$$\cos(\pi-\theta) = -\cos\theta, \qquad e^{im(\phi+\pi)} = (-1)^m e^{im\phi}.$$

Also, associated Legendre functions satisfy: $P_\ell^m(-x) = (-1)^{\ell+m}P_\ell^m(x)$.

Substituting into the expression:

$$\begin{aligned} Y_\ell^m(\pi - \theta, \phi + \pi) &= N_{\ell m} P_\ell^m(-\cos\theta)\,(-1)^m e^{im\phi} \\ &= (-1)^{\ell+m}(-1)^m N_{\ell m} P_\ell^m(\cos\theta)\, e^{im\phi} \\ &= (-1)^\ell Y_\ell^m(\theta, \phi). \end{aligned}$$

4.5. **Spherical harmonics expansion.**

Theorem 4.10. *The Laplace spherical harmonics $Y_\ell^m : \mathbb{S}^2 \to \mathbb{C}$ form a complete orthonormal system in the Hilbert space $L^2(\mathbb{S}^2)$ of square-integrable complex-valued functions on the unit sphere. As such, any function $f \in L^2(\mathbb{S}^2)$ can be expressed as:*

$$f(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_\ell^m \, Y_\ell^m(\theta, \phi),$$

where the coefficients are given by

$$f_\ell^m = \int_{\mathbb{S}^2} f(\theta, \phi)\, Y_\ell^{m*}(\theta, \phi)\, d\Omega = \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi)\, Y_\ell^{m*}(\theta, \phi)\, \sin\theta\, d\theta\, d\phi.$$

This expansion converges in the L^2-norm:

$$\lim_{N \to \infty} \int_0^{2\pi} \int_0^{\pi} \left| f(\theta, \phi) - \sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} f_\ell^m Y_\ell^m(\theta, \phi) \right|^2 \sin\theta\, d\theta\, d\phi = 0.$$

Proof. Let $\{Y_\ell^m\}$ denote the full system of spherical harmonics. These functions satisfy the orthonormality relation 4.3:

$$\int_{\mathbb{S}^2} Y_\ell^m(\theta, \phi)\, Y_{\ell'}^{m'*}(\theta, \phi)\, d\Omega = \delta_{\ell\ell'}\delta_{mm'}.$$

Given $f \in L^2(\mathbb{S}^2)$, we define its projection onto the basis function Y_ℓ^m as:

$$\langle f, Y_\ell^m \rangle = \int_{\mathbb{S}^2} f(\theta, \phi)\, Y_\ell^{m*}(\theta, \phi)\, d\Omega. \tag{4.3}$$

Define the partial sum of the expansion:

$$S_N(\theta, \phi) = \sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} f_\ell^m \, Y_\ell^m(\theta, \phi). \tag{4.4}$$

We now consider the error:

$$\|f - S_N\|_{L^2(\mathbb{S}^2)}^2 = \int_{\mathbb{S}^2} |f(\theta, \phi) - S_N(\theta, \phi)|^2 \, d\Omega.$$

By the Pythagorean identity in Hilbert spaces, and the fact that S_N is the orthogonal projection of f onto the finite-dimensional subspace spanned by $\{Y_\ell^m : \ell \le N\}$, this error is minimized. Since $\{Y_\ell^m\}$ is a complete orthonormal basis, we have: $\lim_{N\to\infty} \|f - S_N\|^2_{L^2(\mathbb{S}^\mathbb{S})} = 0$, which yields the convergence result:

$$\lim_{N\to\infty} \int_0^{2\pi} \int_0^{\pi} \left| f(\theta,\phi) - \sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} f_\ell^m Y_\ell^m(\theta,\phi) \right|^2 \sin\theta \, d\theta \, d\phi = 0. \tag{4.5}$$

Remark 4.11. The appearance of the complex conjugate arises from the definition of the inner product in complex Hilbert spaces, where $\langle f, g\rangle = \int_{\mathbb{S}^2} f(\theta,\phi)\,\overline{g(\theta,\phi)}\, d\Omega$.

Corollary 4.12. *A square-integrable function $f : \mathbb{S}^2 \to \mathbb{R}$ can also be expanded in terms of the real harmonics $Y_{\ell m} : \mathbb{S}^2 \to \mathbb{R}$ as a sum*

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m}\, Y_{\ell m}(\theta,\phi).$$

Theorem 4.13 (Parseval's Identity)**.** *Let $f \in L^2(S^2)$ have the spherical harmonic expansion*

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_\ell^m Y_\ell^m(\theta,\phi), \quad \textit{where} \quad a_\ell^m = \int_{S^2} f(\theta,\phi)\, \overline{Y_\ell^m(\theta,\phi)}\, d\Omega,$$

Then,

$$\int_{S^2} |f(\theta,\phi)|^2 \, d\Omega = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} |a_\ell^m|^2.$$

Proof. We compute:

$$\int_{S^2} |f(\theta,\phi)|^2 \, d\Omega = \int_{S^2} \left| \sum_{\ell,m} a_\ell^m Y_\ell^m(\theta,\phi) \right|^2 d\Omega = \int_{S^2} \left(\sum_{\ell,m} a_\ell^m Y_\ell^m \right) \overline{\left(\sum_{\ell',m'} a_{\ell'}^{m'} Y_{\ell'}^{m'} \right)} \, d\Omega.$$

Expansion of the product gives us:

$$= \sum_{\ell,m} \sum_{\ell',m'} a_\ell^m \overline{a_{\ell'}^{m'}} \int_{S^2} Y_\ell^m \, \overline{Y_{\ell'}^{m'}} \, d\Omega.$$

By orthonormality (4.3), all terms with $(\ell,m) \neq (\ell',m')$ are zero, and only the diagonal terms $(\ell,m) = (\ell',m')$ remain. Hence,

$$\int_{S^2} |f(\theta,\phi)|^2 \, d\Omega = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} |a_\ell^m|^2.$$

5. Further Results on Spherical Harmonics

5.1. Addition Theorem. [1, p.797-800]

Lemma 5.1. *Two angles (θ_1,ϕ_1) and (θ_2,ϕ_2) in the spherical co-ordinate system, are separated by an angle γ, and the satisfy the identity*

$$\cos(\gamma) = \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\cos(\phi_1 - \phi_2)$$

Proof. The Cartesian position vector $\mathbf{r}_1$ is given by$\mathbf{r}_1 = \langle \sin\theta_1 \cos\phi_1,\ \sin\theta_1 \sin\phi_1,\ \cos\theta_1 \rangle$, and the Cartesian position vector $\mathbf{r}_2$, by $\mathbf{r}_2 = \langle \sin\theta_2 \cos\phi_2,\ \sin\theta_2 \sin\phi_2,\ \cos\theta_2 \rangle$. Recall that the dot product of two vectors gives the cosine of the angle between them: $\mathbf{r}_1 \cdot \mathbf{r}_2 = |\mathbf{r}_1||\mathbf{r}_2| \cos\gamma$. Since both are unit vectors, $|\mathbf{r}_1| = |\mathbf{r}_2| = 1$, so $\cos\gamma = \mathbf{r}_1 \cdot \mathbf{r}_2$. Computing the dot product:

$$\begin{aligned}
\mathbf{r}_1 \cdot \mathbf{r}_2 &= (\sin\theta_1 \cos\phi_1)(\sin\theta_2 \cos\phi_2) + (\sin\theta_1 \sin\phi_1)(\sin\theta_2 \sin\phi_2) + (\cos\theta_1)(\cos\theta_2) \\
&= \sin\theta_1 \sin\theta_2[\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2] + \cos\theta_1 \cos\theta_2 \\
&= \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2) + \cos\theta_1 \cos\theta_2.
\end{aligned}$$

Hence,

$$\cos\gamma = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2). \tag{5.1}$$

Theorem 5.2. *Let $\mathbf{r}, \mathbf{r}' \in \mathbb{S}^2$, and let γ (5.1) denote the angle between them, $\cos\gamma = \mathbf{r} \cdot \mathbf{r}'$. Then, for each $n \in \mathbb{N}_0$, the following identity holds:*

$$P_n(\cos\gamma) = \frac{4\pi}{2n+1} \sum_{m=-n}^{n} Y_n^m(\theta, \phi)\, \overline{Y_n^m}(\theta', \phi'),$$

Proof. We follow the inductive approach of outlined in [12], which rewrites both sides in terms of trigonometric expansions and proves equality of coefficients. Let the spherical coordinates of $\mathbf{r}, \mathbf{r}' \in \mathbb{S}^2$ be $\mathbf{r} = (\theta, \phi), \quad \mathbf{r}' = (\theta', \phi')$. Define $t = \theta$, $T = \theta'$, and let $\Phi = \phi - \phi'$. Then, the spherical angle γ between $\mathbf{r}$ and $\mathbf{r}'$ satisfies (5.1):

$$\cos\gamma = \cos t \cos T + \sin t \sin T \cos\Phi.$$

Base cases: For $n = 0$, we have $P_0(x) = 1$, and the RHS becomes:

$$\frac{4\pi}{1} \cdot Y_0^0(\theta, \phi)\overline{Y_0^0}(\theta', \phi') = 4\pi \cdot \frac{1}{4\pi} = 1.$$

For $n = 1$, $P_1(x) = x$, and using equation (5.1), we have:

$$P_1(\cos\gamma) = \cos\gamma = \cos t \cos T + \sin t \sin T \cos\Phi,$$

which matches the sum over $Y_1^m \overline{Y_1^m}$.
Inductive hypothesis: Suppose the theorem holds for degrees $n-1$ and n, i.e.,

$$P_n(\cos\gamma) = \sum_{k=0}^{n} A_k \cos(k\Phi), \quad P_{n-1}(\cos\gamma) = \sum_{k=0}^{n-1} B_k \cos(k\Phi),$$

where A_k, B_k depend on t and T.
Recurrence: Use the Legendre polynomial recurrence:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x). \tag{5.2}$$

Substituting $x = \cos\gamma$ in (5.2) and using (5.1), we get:

$$P_{n+1}(\cos\gamma) = \frac{1}{n+1}\left[(2n+1)\cos\gamma \cdot P_n(\cos\gamma) - nP_{n-1}(\cos\gamma)\right]. \tag{5.3}$$

We expand $P_n(\cos\gamma)$ and $P_{n-1}(\cos\gamma)$ into trigonometric polynomials in $\cos(k\Phi)$, and observe that multiplying $\cos\gamma$ (which contains $\cos\Phi$) with $\cos(k\Phi)$ yields terms of the form $\cos((k \pm 1)\Phi)$ via:

$$\cos\Phi \cdot \cos(k\Phi) = \frac{1}{2}\left[\cos((k+1)\Phi) + \cos((k-1)\Phi)\right]. \tag{5.4}$$

Thus, $\cos\gamma \cdot P_n(\cos\gamma)$ produces a trigonometric polynomial of degree $n+1$, and so does the recurrence (5.3). We write:

$$P_{n+1}(\cos\gamma) = \sum_{k=0}^{n+1} A_k^{(n+1)} \cos(k\Phi),$$

and compute each $A_k^{(n+1)}$ explicitly by collecting coefficients in the expansion from (5.3), using the identities above. In particular:

- $A_0^{(n+1)}$ arises from the constant terms.
- $A_k^{(n+1)}$ for $1 \le k \le n$ arise from two contributions via (5.4).
- $A_{n+1}^{(n+1)}$ arises only from the highest-degree term in the expansion.

Algebraic manipulation (as done in [12], Section 4) shows that each $A_k^{(n+1)}$ matches the expected coefficient in the Fourier representation of:

$$P_{n+1}\cos(\gamma) = \frac{4\pi}{2n+3} \sum_{m=-n-1}^{n+1} Y_{n+1}^m(\theta,\phi)\overline{Y_{n+1}^m}(\theta',\phi').$$

Conclusion: Since the expansion for $P_{n+1}(\cos\gamma)$ matches the spherical harmonics sum, the theorem holds for $n+1$. By induction,

$$P_n(\cos\gamma) = \frac{4\pi}{2n+1} \sum_{m=-n}^{n} Y_n^m(\theta,\phi)\,\overline{Y_n^m}(\theta',\phi'), \quad \forall n \in \mathbb{N}_0$$

.

5.2. **Zonal Harmonics and Kernel Representations.** [3] Let $\mathcal{H}_\ell$ denote the space of spherical harmonics of fixed degree ℓ on $\mathbb{S}^2$, of dimension $2\ell+1$. For any function $f \in \mathcal{H}_\ell$, we can express f in terms of an integral against a kernel:

$$f(\mathbf{x}) = \int_{\mathbb{S}^2} K_\ell(\mathbf{x},\mathbf{x}') f(\mathbf{x}')\, d\Omega'. \tag{5.5}$$

This equation defines the *reproducing kernel property*, and $K_\ell(\mathbf{x},\mathbf{x}')$ is called a *kernel*—a function of two points on the sphere that acts as a symmetric "interpolator" across the space.

5.2.1. *Zonal Harmonics.* A *zonal harmonic* is a spherical harmonic symmetric about an axis, typically taken as the z-axis. These correspond to the $m=0$ harmonics:

$$Z_\ell(\theta) := Y_\ell^0(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta), \tag{5.6}$$

where P_ℓ is the Legendre polynomial of degree ℓ. Zonal harmonics depend only on θ, the angle from the z-axis, and are independent of ϕ, making them axisymmetric.

The significance of zonal harmonics increases when they are rotated so that their axis of symmetry points through an arbitrary point $\mathbf{x}' \in \mathbb{S}^2$. The rotated version defines a function of a second point $\mathbf{x}$, depending only on the angle $\gamma = \angle(\mathbf{x}, \mathbf{x}')$. The result is the kernel:

$$K_\ell(\mathbf{x}, \mathbf{x}') := \sum_{m=-\ell}^{\ell} Y_\ell^m(\mathbf{x})\overline{Y_\ell^m(\mathbf{x}')} = \frac{2\ell+1}{4\pi} P_\ell(\cos\gamma). \tag{5.7}$$

This is a *zonal kernel* centered at $\mathbf{x}'$ and evaluated at $\mathbf{x}$, and is itself a zonal harmonic in $\mathbf{x}$ around $\mathbf{x}'$. The right-hand side depends only on γ and not on any azimuthal component.

5.2.2. *Reproducing Property.* Let $f \in \mathcal{H}_\ell$ be expanded as:

$$f(\mathbf{x}') = \sum_{m=-\ell}^{\ell} a_m Y_\ell^m(\mathbf{x}').$$

Then:

$$\begin{aligned}\int_{\mathbb{S}^2} K_\ell(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\Omega' &= \sum_{m=-\ell}^{\ell} Y_\ell^m(\mathbf{x}) \int_{\mathbb{S}^2} \overline{Y_\ell^m(\mathbf{x}')} f(\mathbf{x}')\, d\Omega' \\ &= \sum_{m=-\ell}^{\ell} a_m Y_\ell^m(\mathbf{x}) = f(\mathbf{x}).\end{aligned}$$

This confirms that K_ℓ is a *reproducing kernel* for $\mathcal{H}_\ell$. This property expresses $f(\mathbf{x})$ as a *weighted average of zonal harmonics* centered at all $\mathbf{x}'$, and the weight at each $\mathbf{x}'$ is given by the value $f(\mathbf{x}')$.

5.2.3. *Projection via Kernel.* The kernel K_ℓ defines a projection operator $\mathbb{P}_\ell : L^2(\mathbb{S}^2) \to \mathcal{H}_\ell$ given by:

$$(\mathbb{P}_\ell f)(\mathbf{x}) := \int_{\mathbb{S}^2} K_\ell(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\Omega'.$$

This operator projects any square-integrable function onto its degree-ℓ component.
Let $g \in \mathcal{H}_\ell$. Then for any $f \in L^2(\mathbb{S}^2)$,

$$\begin{aligned}\langle \mathbb{P}_\ell f, g \rangle &= \int_{\mathbb{S}^2} \left(\int_{\mathbb{S}^2} K_\ell(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\Omega' \right) \overline{g(\mathbf{x})}\, d\Omega \\ &= \int_{\mathbb{S}^2} f(\mathbf{x}') \left(\int_{\mathbb{S}^2} K_\ell(\mathbf{x}, \mathbf{x}') \overline{g(\mathbf{x})}\, d\Omega \right) d\Omega' \\ &= \int_{\mathbb{S}^2} f(\mathbf{x}') \overline{g(\mathbf{x}')}\, d\Omega' = \langle f, g \rangle.\end{aligned}$$

So $\mathbb{P}_\ell$ is an *orthogonal projection* onto $\mathcal{H}_\ell$, and again this is achieved using only *zonal kernels.*

5.2.4. *Generalization.* Given any $f \in L^2(\mathbb{S}^2)$, the projection $\mathbb{P}_\ell f$ is the best degree-ℓ approximation to f in the L^2 norm, and is given by integration against the zonal kernel K_ℓ. That is, every f can be expressed as:

$$f(\mathbf{x}) = \sum_{\ell=0}^{\infty} \int_{\mathbb{S}^2} K_\ell(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\Omega',$$

where each term in the sum is a zonal harmonic (via K_ℓ) integrated against f. This representation shows that arbitrary functions on the sphere can be reconstructed as infinite combinations of zonal harmonics centered at different points.

6. Generalization to $\mathbb{S}^n$: Hyperspherical Harmonics

In this section, we extend the theory of spherical harmonics from $\mathbb{S}^2$ to the general n-dimensional unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. We develop the function spaces and coordinate systems, derive the Laplace–Beltrami Operator on $\mathbb{S}^n$, and study its eigenfunctions(hyperspherical harmonics) along with their eigenvalues, orthogonality, and expansion properties. [8, 9]

6.1. Function Spaces and Geometry.

We denote by $\mathbb{S}^n$ the n-dimensional unit sphere in $\mathbb{R}^{n+1}$, $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. Let $L^2(\mathbb{S}^n)$ denote the Hilbert space of complex-valued square-integrable functions $f : \mathbb{S}^n \to \mathbb{C}$, with respect to the standard surface measure $d\Omega_n$ on $\mathbb{S}^n$:

$$L^2(\mathbb{S}^n) = \left\{ f : \mathbb{S}^n \to \mathbb{C} \;\middle|\; \int_{\mathbb{S}^n} |f(x)|^2 \, d\Omega_n(x) < \infty \right\}.$$

The inner product on $L^2(\mathbb{S}^n)$ is given by

$$\langle f, g \rangle = \int_{\mathbb{S}^n} f(x)\overline{g(x)} \, d\Omega_n(x),$$

and the associated norm is $\|f\| = \sqrt{\langle f, f \rangle}$. A set $\{Y_\ell\}$ of functions on $\mathbb{S}^n$ is orthonormal if $\langle Y_\ell, Y_k \rangle = \delta_{\ell k}$, and complete if their span is dense in $L^2(\mathbb{S}^n)$.

The standard coordinates on $\mathbb{S}^n$ are the hyperspherical coordinates $(\theta_1, \theta_2, \ldots, \theta_{n-1}, \phi)$, where $\theta_1, \ldots, \theta_{n-1} \in [0, \pi]$, and $\phi \in [0, 2\pi)$. In these coordinates, a point $x \in \mathbb{S}^n$ is represented as:

$$\begin{aligned}
x_1 &= \cos\theta_1, \\
x_2 &= \sin\theta_1 \cos\theta_2, \\
x_3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3, \\
&\vdots \\
x_n &= \sin\theta_1 \cdots \sin\theta_{n-2} \cos\theta_{n-1}, \\
x_{n+1} &= \sin\theta_1 \cdots \sin\theta_{n-2} \sin\theta_{n-1} \cos\phi, \\
x_{n+2} &= \sin\theta_1 \cdots \sin\theta_{n-2} \sin\theta_{n-1} \sin\phi.
\end{aligned}$$

The volume element (surface measure) in these coordinates is:

$$d\Omega_n = \left(\prod_{j=1}^{n-1} (\sin\theta_j)^{n-j-1} \right) d\theta_1 \cdots d\theta_{n-1} \, d\phi.$$

6.2. Laplacian and Laplace–Beltrami Operator.

We begin with the standard Laplacian in Euclidean space $\mathbb{R}^{n+1}$. Given a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, the Laplacian is defined by (the divergence of the gradient)

$$\Delta f = \nabla \cdot (\nabla f) = \sum_{i=1}^{n+1} \frac{\partial^2 f}{\partial x_i^2}.$$

However, when the domain is constrained to the surface of a manifold such as $\mathbb{S}^n$, this Euclidean formulation fails to account for intrinsic curvature (`Section 3.1`). To adapt the Laplacian to curved spaces, we replace the Euclidean gradient and divergence with their intrinsic counterparts defined using the Riemannian metric. We thus obtain the Laplace–Beltrami operator, denoted $\Delta_{\mathbb{S}^n}$, which generalizes Δ to arbitrary Riemannian manifolds. In hyperspherical coordinates $(\theta_1, \dots, \theta_{n-1}, \phi)$ on $\mathbb{S}^n$, the Laplace–Beltrami operator takes the recursive form:

$$\Delta_{\mathbb{S}^n} f = \frac{1}{\sin^{n-1}\theta_1}\frac{\partial}{\partial\theta_1}\left(\sin^{n-1}\theta_1 \frac{\partial f}{\partial\theta_1}\right) + \frac{1}{\sin^2\theta_1}\Delta_{\mathbb{S}^{n-1}} f. \tag{6.1}$$

This identity reflects the spherical symmetry of $\mathbb{S}^n$ and allows the operator to be separated recursively. At each stage, the angular coordinate θ_j contributes a radial-type term and a rescaled Laplace–Beltrami operator on a lower-dimensional sphere. The base case is $\Delta_{\mathbb{S}^1} = \frac{\partial^2}{\partial\phi^2}$. This recursive structure is essential for solving the eigenvalue problem on $\mathbb{S}^n$ via separation of variables, which we develop in the next subsection.

6.3. **Eigenvalue Problem.** We now consider the eigenvalue problem for the Laplace–Beltrami operator on the unit n-sphere $\mathbb{S}^n$. Given a smooth function $f : \mathbb{S}^n \to \mathbb{C}$, we seek all solutions to the equation $\Delta_{\mathbb{S}^n} f + \lambda f = 0$, where $\lambda \in \mathbb{R}$ is an eigenvalue, and $f \not\equiv 0$ is the corresponding eigenfunction.

The solutions to this eigenvalue problem are called *hyperspherical harmonics*. They generalize the classical spherical harmonics Y_ℓ^m on $\mathbb{S}^2$ to higher dimensions. For each integer $\ell \geq 0$, there exists a finite-dimensional eigenspace $\mathcal{H}_\ell(\mathbb{S}^n)$ of homogeneous harmonic polynomials of degree ℓ restricted to the sphere. The dimension of this space depends on n and ℓ, and the corresponding eigenvalue is $\lambda_\ell = \ell(\ell + n - 1)$.

Thus, hyperspherical harmonics are defined as the eigenfunctions $Y_\ell^{(n)} : \mathbb{S}^n \to \mathbb{C}$ satisfying

$$\Delta_{\mathbb{S}^n} Y_\ell^{(n)} = -\ell(\ell + n - 1) Y_\ell^{(n)}.$$

These functions are orthogonal with respect to the inner product on $L^2(\mathbb{S}^n)$ and form a complete set over the space. For fixed ℓ, the functions $\{Y_{\ell,i}^{(n)}\}_{i=1}^{d_\ell}$ form an orthonormal basis for the eigenspace $\mathcal{H}_\ell(\mathbb{S}^n)$, where $d_\ell = \dim \mathcal{H}_\ell(\mathbb{S}^n)$. In the next subsection, we derive the eigenvalues and compute this dimension explicitly.

6.4. **Eigenvalues and Eigenspace.** We now state two fundamental results concerning the structure of these eigenspaces, however proofs are omitted in the scope of this paper.

Proposition 6.1 (Eigenvalues of the Laplace–Beltrami Operator)**.** *Let $Y_\ell^{(n)}$ be a hyperspherical harmonic of degree ℓ on $\mathbb{S}^n$. Then it satisfies the eigenvalue equation $\Delta_{\mathbb{S}^n} Y_\ell^{(n)} = -\ell(\ell + n - 1) Y_\ell^{(n)}$,for each integer $\ell \geq 0$.*

The integer ℓ corresponds to the degree of the harmonic polynomial in $\mathbb{R}^{n+1}$ from which the hyperspherical harmonic is obtained by restriction to the unit sphere. Each value of ℓ defines a distinct eigenspace of the Laplace–Beltrami operator.

Proposition 6.2. *The space $\mathcal{H}_\ell(\mathbb{S}^n)$ of degree-ℓ hyperspherical harmonics has dimension*

$$\dim \mathcal{H}_\ell(\mathbb{S}^n) = \frac{(2\ell + n - 1)(\ell + n - 2)!}{\ell!(n-1)!}.$$

This formula counts the number of linearly independent eigenfunctions corresponding to the eigenvalue $\ell(\ell + n - 1)$. In the case $n = 2$, the expression simplifies to $\dim \mathcal{H}_\ell(\mathbb{S}^2) = 2\ell + 1$, which matches the (familiar) count of 2-sphere spherical harmonics Y_ℓ^m. These eigenspaces are mutually orthogonal in $L^2(\mathbb{S}^n)$ and together span the space. Thus, the hyperspherical harmonics form a complete orthonormal system for square-integrable functions on the sphere.

6.5. $\mathbb{S}^3$**.** We now examine the case $n = 3$, corresponding to the 3-dimensional unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4$. The eigenvalue formula simplifies to

$$\lambda_\ell = \ell(\ell + 2),$$

and the dimension of the eigenspace becomes

$$\dim \mathcal{H}_\ell(\mathbb{S}^3) = (\ell + 1)^2.$$

This structure is particularly transparent for small values of ℓ:

ℓ	Eigenvalue λ_ℓ	Dimension $\dim \mathcal{H}_\ell(\mathbb{S}^3)$
0	0	1
1	3	4
2	8	9
3	15	16
4	24	25

Each eigenspace $\mathcal{H}_\ell(\mathbb{S}^3)$ contains $(\ell + 1)^2$ mutually orthogonal hyperspherical harmonics of degree ℓ. These form a complete orthonormal basis for $L^2(\mathbb{S}^3)$.

Proposition 6.3. *Let $Y^{(3)}_{\ell,i}$ and $Y^{(3)}_{k,j}$ be hyperspherical harmonics on $\mathbb{S}^3$. Then*

$$\int_{\mathbb{S}^3} Y^{(3)}_{\ell,i}(\xi)\overline{Y^{(3)}_{k,j}(\xi)}\, d\Omega_3(\xi) = \delta_{\ell k}\delta_{ij}.$$

As a simple verification, consider the constant function $Y^{(3)}_0$, which spans the degree-zero eigenspace. To normalize it, we compute the volume of $\mathbb{S}^3$ [10]:

$$\mathrm{Vol}(\mathbb{S}^n) = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \quad \text{so} \quad \mathrm{Vol}(\mathbb{S}^3) = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2.$$

Hence, the normalized constant eigenfunction is

$$Y^{(3)}_0 = \frac{1}{\sqrt{2\pi^2}}.$$

We now verify orthonormality:

$$\int_{\mathbb{S}^3} \left(\frac{1}{\sqrt{2\pi^2}}\right)^2 d\Omega_3 = \frac{1}{2\pi^2} \cdot 2\pi^2 = 1.$$

The $\mathbb{S}^3$ case offers a concrete illustration of the general theory. The eigenvalues grow quadratically in ℓ, while the dimension of each eigenspace grows as a perfect square. These harmonics are used extensively in applications involving three-dimensional rotations, wave equations, and quantum systems with spherical symmetry.

6.6. **Orthonormality of Hyperspherical Harmonics.** The hyperspherical harmonics $\{Y^{(n)}_{\ell,i}\}$ form an orthonormal basis for the Hilbert space $L^2(\mathbb{S}^n)$ with respect to the standard inner product:

$$\langle f, g\rangle = \int_{\mathbb{S}^n} f(\xi)\overline{g(\xi)}\, d\Omega_n(\xi).$$

We now state the orthonormality property precisely.

Theorem 6.4 (Orthonormality). *Let $\ell, k \in \mathbb{Z}_{\geq 0}$ and let $\{Y_{\ell,i}^{(n)}\}_{i=1}^{d_\ell}$ and $\{Y_{k,j}^{(n)}\}_{j=1}^{d_k}$ be orthonormal bases for the eigenspaces $\mathcal{H}_\ell(\mathbb{S}^n)$ and $\mathcal{H}_k(\mathbb{S}^n)$. Then*

$$\int_{\mathbb{S}^n} Y_{\ell,i}^{(n)}(\xi)\,\overline{Y_{k,j}^{(n)}(\xi)}\,d\Omega_n(\xi) = \delta_{\ell k}\delta_{ij}.$$

Sketch of Proof. The Laplace–Beltrami operator is self-adjoint on the compact manifold $\mathbb{S}^n$ with respect to the L^2 inner product. This implies that eigenfunctions corresponding to distinct eigenvalues are orthogonal. Since each $\mathcal{H}_\ell(\mathbb{S}^n)$ is a finite-dimensional eigenspace, its elements can be orthonormalized using the Gram–Schmidt process. The result then follows directly.

6.7. **Expansion theorem.** Just as classical Fourier series decompose functions into orthogonal trigonometric modes, any square-integrable function on the sphere can be decomposed into hyperspherical harmonics. This yields a complete orthogonal expansion.

Theorem 6.5 (Expansion Theorem). *Let $f \in L^2(\mathbb{S}^n)$. Then f can be written as an infinite sum of hyperspherical harmonics:*

$$f(\xi) = \sum_{\ell=0}^{\infty}\sum_{i=1}^{d_\ell}\langle f, Y_{\ell,i}^{(n)}\rangle\, Y_{\ell,i}^{(n)}(\xi),$$

where $d_\ell = \dim \mathcal{H}_\ell(\mathbb{S}^n)$ and the series converges in L^2.

Sketch of Proof. The Laplace–Beltrami operator has a countable orthonormal basis of eigenfunctions in $L^2(\mathbb{S}^n)$ due to compactness and self-adjointness. Since the hyperspherical harmonics form an orthonormal basis for each eigenspace $\mathcal{H}_\ell(\mathbb{S}^n)$, their union spans $L^2(\mathbb{S}^n)$, and any function in this space can be expanded as a sum over them.

Theorem 6.6 (Parseval's Identity). *Let $f \in L^2(\mathbb{S}^n)$ with expansion coefficients*

$$a_{\ell,i} = \langle f, Y_{\ell,i}^{(n)}\rangle.$$

Then

$$\|f\|^2 = \int_{\mathbb{S}^n} |f(\xi)|^2\,d\Omega_n(\xi) = \sum_{\ell=0}^{\infty}\sum_{i=1}^{d_\ell}|a_{\ell,i}|^2.$$

Sketch of Proof. This identity follows directly from the orthonormality of the basis and the completeness of the expansion. It is a special case of Bessel's inequality, which becomes an equality when the expansion is over a complete orthonormal set.

Conclusion

This exposition established the foundational theory of spherical harmonics via Laplace's equation on the sphere. We developed their construction, orthogonality, expansion properties, and generalization to $\mathbb{S}^n$. These structures form analytical tools for understanding symmetry, solving boundary value problems or decomposing functions on curved spaces. For further readings, one can refer to the following [2], [6], [7], [4], [5], or go into greater depths of other citations that I just touched upon at the surface level.

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