Matrix Tree Theorem & The Number of Spanning Trees

Omar Mohamed Said

IRPW

July 10, 2025

Outline

The Problem: Counting Spanning Trees

The Challenge

Counting the number of spanning trees by hand is boring, and for bigger graphs it becomes frustrating.

The Solution

The **Matrix-Tree Theorem** provides a powerful and elegant algebraic method to solve this problem, revealing connections between graph theory and linear algebra.

Basic Definitions

Graph

A simple graph G is an ordered pair (V, E), where V is a finite set of vertices, and E is a set of unordered pairs of distinct vertices, called edges.

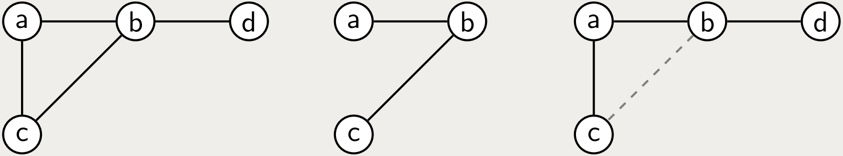
Degree of a Vertex

The degree of a vertex v, denoted $\deg(v)$, is the number of edges incident to it.

Tree and Spanning Tree

A **tree** is a connected acyclic graph. A **spanning tree** of a connected graph G is a subgraph that is a tree and includes all vertices of G. The number of spanning trees is denoted by $\tau(G)$.

Basic definitions



(a) A simple graph G_1. (b) A subgraph of G_1. (c) Spanning tree T_1.

Figure: Illustrations of graph theory concepts.

Key Graph Matrices

Adjacency Matrix (A)

An $n \times n$ matrix where $A_{ij} = 1$ if vertices v_i and v_j are connected by an edge, and 0 otherwise. It represents the connections in the graph.

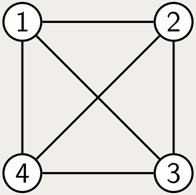
Degree Matrix (D)

An $n \times n$ diagonal matrix where the entry D_{ii} is the degree of vertex v_i (the number of edges connected to it). All off-diagonal entries are 0.

Incidence Matrix (I)

An $n \times m$ matrix (for n vertices and m edges) that describes how vertices and edges are connected. For an arbitrarily oriented edge $e_k = (v_i, v_j)$, the entries are $I_{ik} = 1$, $I_{jk} = -1$, and 0 otherwise.

Matrices of the Complete Graph K_4



Degree (D) **Adjacency (A)** **Incidence (I)**

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$$

The Laplacian Matrix

Laplacian Matrix (L)

The Laplacian is defined as the Degree Matrix minus the Adjacency Matrix:

$$L = D - A$$

It is a fundamental matrix in spectral graph theory and is central to the Matrix-Tree Theorem.

Laplacian Matrix $L = D - A$

$$L_{K_4} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

The Matrix-Tree Theorem

Theorem Statement

Let G be a connected graph with n vertices and let L be its Laplacian matrix. The number of spanning trees of G, denoted by $\tau(G)$, is equal to any cofactor of L.

$$\tau(G) = (-1)^{i+j} \det(L_{ij})$$

where L_{ij} is the sub-matrix of L obtained by deleting row i and column j.

Number of spanning trees via the Matrix-Tree theorem

The Laplacian matrix for K_4 is:

$$L_{K_4} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

Removing the first row and column:

$$L'_{K_4} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

$\det(L'_{K_4}) = 16.$

So, the Matrix-Tree Theorem predicts $\tau(K_4) = 16$.

The 16 Spanning Trees of K_4

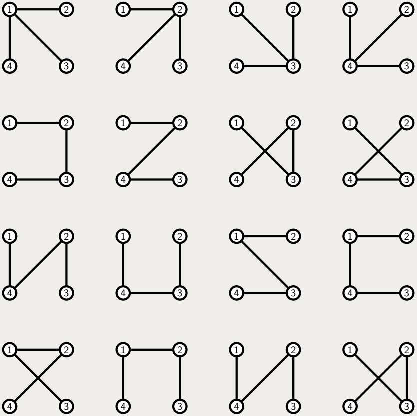


Figure: By enumeration, we confirm there are 16 spanning trees.

A Formula for Cycles in a Graph

Theorem

For any connected graph, the number of vertices (V), edges (E), and independent cycles (C) are related by the formula:

$$V - E + C = 1$$

Proof Sketch

1. Start with a spanning tree of the graph. By definition, it has V vertices and $V - 1$ edges, and no cycles.
2. The remaining edges in the graph that are not in the spanning tree number $E - (V - 1)$.
3. Each of these remaining edges, when added back to the spanning tree, creates exactly one independent cycle.
4. Therefore, the number of cycles is $C = E - (V - 1)$. Rearranging this gives the theorem: $V - E + C = 1$.

The Cauchy-Binet Formula

The Cauchy-Binet Theorem

Let A be an $m \times n$ matrix, and let B be an $n \times m$ matrix. If $m > n$, then $\det(AB) = 0$. If $m \leq n$, then

$$\det(AB) = \sum_{S} (\det A[S])(\det B[S])$$

where S ranges over all m-element subsets of the columns of A and rows of B.

Example Setup

Let A be a 2×3 matrix and B be a 3×2 matrix.

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \quad B = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix}$$

Cauchy-Binet: Example Calculation

Expanding the Determinant

The determinant $\det(AB)$ is the sum of the products of the determinants of all corresponding 2×2 sub-matrices:

$$\det(AB) = \overbrace{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}}^{\text{Cols 1 \& 2}} + \overbrace{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \cdot \begin{vmatrix} c_1 & d_1 \\ c_3 & d_3 \end{vmatrix}}^{\text{Cols 1 \& 3}} + \overbrace{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \cdot \begin{vmatrix} c_2 & d_2 \\ c_3 & d_3 \end{vmatrix}}^{\text{Cols 2 \& 3}}$$

Proof Strategy

Goal

We want to prove that any cofactor of the Laplacian matrix, such as $\det(L_{11})$, is equal to the number of spanning trees, $\tau(G)$.

Key Term: The Adjugate Matrix

The adjugate of a matrix M, written $\text{adj}(M)$, is the transpose of its cofactor matrix. Its main property is:

$$M \cdot \text{adj}(M) = \det(M) \cdot I$$

Our Two-Step Strategy

1. Show that $\text{adj}(L)$ must be a constant matrix: $\text{adj}(L) = \alpha \cdot J$, where J is the all-ones matrix. This means all cofactors are equal to some constant α.
2. Use the Cauchy-Binet formula to find this constant and show that $\alpha = \tau(G)$.

Proof

From the definition of the adjugate, we have:

$$L \cdot \text{adj}(L) = \det(L) \cdot I$$

Using Properties of the Laplacian

- For a connected graph with n vertices, we know $\text{rank}(L) = n - 1$ so $\det(L) = 0$.
- Therefore, the equation becomes: $L \cdot \text{adj}(L) = 0$.

The equation $L \cdot \text{adj}(L) = 0$ means that every column of $\text{adj}(L)$ is in the null space of L. For a connected graph, this space is simply all multiples of the all-ones vector $\mathbf{1}$. So, every column of $\text{adj}(L)$ has the same value α in every entry, proving that $\text{adj}(L) = \alpha \cdot J$.

Proof

Finding α with a Cofactor

Since all cofactors are equal to α, we only need to calculate one. Let's find the $(1,1)$-cofactor:

$$\alpha = \det(L_{11})$$

We know $L = II^T$, so we can write $L_{11} = I_1 I_1^T$, where I_1 is the incidence matrix with the first row removed.

Applying the Cauchy-Binet Formula

$$\det(L_{11}) = \det(I_1 I_1^T) = \sum_{F \subseteq E, |F| = n-1} (\det((I_1)_F))^2$$

And $\det((I_1)_F)$ is ± 1 if the edges in F form a spanning tree, and 0 otherwise.

$$\alpha = \tau(G)$$

Identity: L = I I^T

Proof

1. **Diagonal Entries ($i = j$):** The entry $(II^T)_{ii}$ is the dot product of the i-th row of I with itself:

$$(II^T)_{ii} = \sum_{k=1}^{m} I_{ik}^2$$

Since I_{ik} is ± 1 only if edge k touches vertex i, I_{ik}^2 is 1 for every incident edge and 0 otherwise. The sum is simply the number of edges connected to vertex i, which is $\deg(v_i)$. This matches L_{ii}.

Identity: $L = I\, I^T$

Proof

2. **Off-Diagonal Entries ($i \neq j$):** The entry $(II^T)_{ij}$ is the dot product of rows i and j:

$$(II^T)_{ij} = \sum_{k=1}^{m} I_{ik} I_{jk}$$

This product is non-zero only if an edge k connects vertices i and j. For that edge, one entry (e.g., I_{ik}) will be $+1$ and the other (I_{jk}) will be -1. Their product is -1, which matches L_{ij}. If no edge connects i and j, the sum is 0.

Cayley's Formula for Complete Graphs

Theorem (Cayley, 1889)

The number of spanning trees of a complete graph K_n is n^{n-2}.

Example: K_6

Using the Matrix-Tree theorem, $\tau(K_6) = 6^{6-2} = 6^4 = 1296$.

Proof Idea

Establish a bijection between labeled trees on n vertices and sequences of length $n-2$ from $\{1, \ldots, n\}$ using **Prüfer codes**.

Prüfer Code: Encoding Example

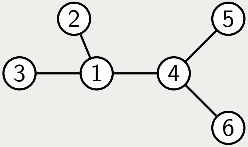


Figure: A labeled tree on 6 vertices.

The Prüfer code is generated by iteratively removing the smallest leaf and recording its neighbor.

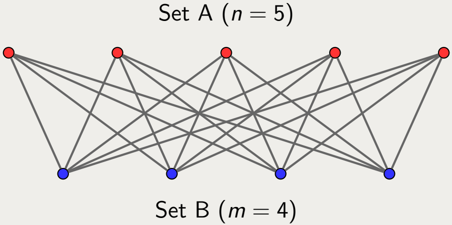
Step (i)	Smallest Leaf (b_i)	Neighbor (a_i)	Remaining Vertices
1	2	1	$\{1, 3, 4, 5, 6\}$
2	3	1	$\{1, 4, 5, 6\}$
3	1	4	$\{4, 5, 6\}$
4	5	4	$\{4, 6\}$

The resulting Prüfer code is the sequence of neighbors: $(\mathbf{1}, \mathbf{1}, \mathbf{4}, \mathbf{4})$.

Other Graph Types: Complete Bipartite Graphs

Definition: Complete Bipartite Graph ($K_{n,m}$)

A bipartite graph has its vertices divided into two disjoint sets, A (with n vertices) and B (with m vertices). Every vertex in set A is connected to every vertex in set B.



Spanning Trees in $K_{n,m}$

Theorem

The number of spanning trees in a complete bipartite graph $K_{n,m}$ is given by the formula:

$$\tau(K_{n,m}) = n^{m-1} m^{n-1}$$

Checking for $K_{5,4}$

For the graph in the paper ($K_{5,4}$), the number of spanning trees would be:

$$\tau(K_{5,4}) = 5^{4-1} \cdot 4^{5-1} = 5^3 \cdot 4^4 = 125 \cdot 256 = 32,000$$

Proof for $K_{n,m}$

Strategy: Using the Eigenvalue Formula

We use the version of the Matrix-Tree Theorem that relates the number of spanning trees to the non-zero eigenvalues (λ_i) of the Laplacian matrix:

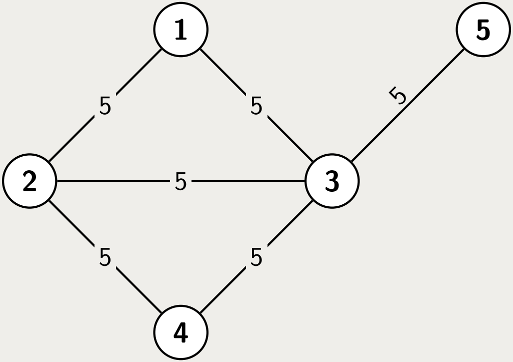
$$\tau(G) = \frac{1}{\text{number of vertices}} \prod \lambda_i$$

For $K_{n,m}$, the number of vertices is $n + m$.

$$\tau(K_{n,m}) = \frac{1}{n+m} \cdot (n+m)^1 \cdot n^{m-1} \cdot m^{n-1}$$

$$\tau(K_{n,m}) = n^{m-1} m^{n-1}$$

A Question on Weighted Graphs



Question

Consider this graph where every edge has a uniform weight of 5.
Is the number of spanning trees in this weighted graph 5 times the number in its unweighted version?

Thank You!

Questions?