An Exposition to Fourier Analysis

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Motivation and Historical Background

Jean-Baptiste Joseph Fourier was the individual to discover the Fourier Series and many of their resulting properties. This was done in search of a solution to the heat equation. He claimed that any arbitrary function could be represented as a series of sines and cosines. Although this claim would later be corrected by major mathematicians such as Dirilecht and Gauss, it set the foundation for the entire field of Fourier Analysis. This field has produced many interesting results —with varying degrees of practicality —and we aim to study some of these results today.

Fourier Series

If we take a periodic function f, we can define its Fourier Series as

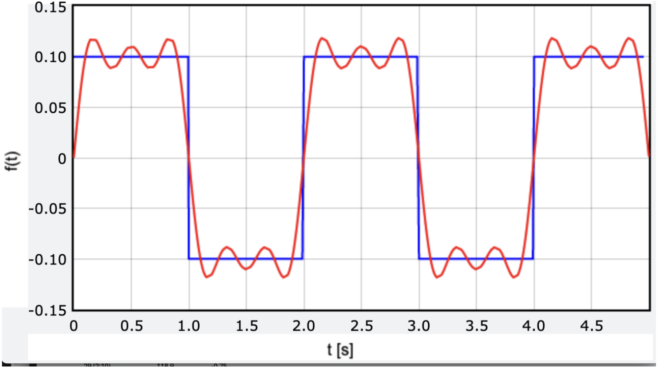
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \tag{0.1}$$

Where we call a_n, b_n and a_0 its Fourier Coefficients. a_n and b_n are the coefficients of the sine and cosine functions respectively, and a_0 is the average value of the function over the interval $[-L, L]$, where here, the interval is $[-\pi, \pi]$.

Remark

While this definition works on the period 2π, for a function defined on $[-p, p]$, there does exist a scaling factor, $\frac{n\pi}{p}$ in order to adjust the oscillating terms to match the interval.

Fourier Series



The classic Fourier Series approximation to the Square Wave.

Orthogonality

It is well known that if two vectors $\vec{v}$ and $\vec{w}$ have a dot product of zero, then they are orthogonal. That is,

$$\vec{v} \cdot \vec{w} = 0 \quad \Rightarrow \quad \vec{v} \perp \vec{w}.$$

In a similar manner, two functions are orthogonal if the integral of their product is zero over a given interval.

Definition

Two functions $f(x)$ and $g(x)$ are **orthogonal** on an interval $[a, b]$ if:

$$\int_a^b f(x)g(x)\,dx = 0$$

Orthogonality

Definition

The system

$$\mathcal{T} := \{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \ldots\}$$

is a complete orthogonal system on $[-\pi, \pi]$. To show the orthogonality of this system, one needs to show that the following integrals vanish:

$$\begin{aligned} \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx &= 0, \qquad m, n \geq 0,\; m \neq n \\ \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx &= 0, \qquad m, n \geq 1,\; m \neq n \\ \int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx &= 0, \qquad m \geq 0,\; n \geq 1 \end{aligned}$$

Coefficients

We can compute the coefficients a_n and b_n using the orthogonality of sine and cosine:

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

To solve for a_n, multiply both sides by $\cos(mx)$, where $m > 1$ and integrate over $[-\pi, \pi]$:
Referencing back to our orthogonality relations, we see that all terms vanish, except for when $n = m$

$$\int_{-\pi}^{\pi} f(x) \cos(mx)\, dx = a_n \int_{-\pi}^{\pi} \cos^2(mx)\, dx = a_m \pi$$

1

[1] Adapted from: Lecture Notes 12.1, Dr. Gabado

Coefficients

So the coefficient is:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx$$

Similarly, for b_n:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx$$

Convergence

Definition (Point wise Convergence)

A sequence of functions $f_n(x)$ converges pointwise to $f(x)$ if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for each } x \in D$$

Definition (Uniform Convergence)

A sequence of functions $f_n(x)$ converges uniformly to $f(x)$ if

$$\sup_{x\in D} |f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty$$

Gibbs' Phenomenon

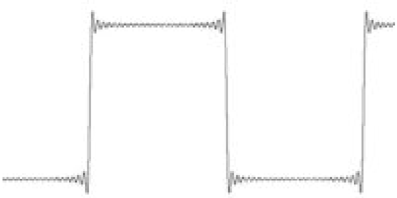
Definition (Gibbs' Phenomenon)

Take f, a function that is piecewise continuous, but has a jump discontinuity. The Nth partial Fourier sum of the function forms large peaks around the jump, both overestimating and underestimating the function. As N increases, the approximation error approaches 9%

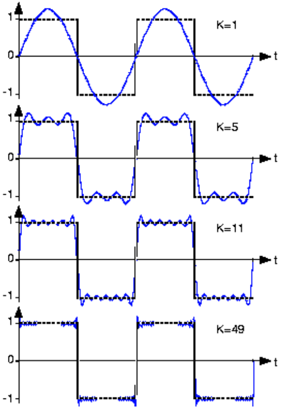
Remark

However, this is not an error when taking the infinite sum of a Fourier Series, which converges to the function at all points of continuity, and to the average of the left and right hand limits at our jump discontinuity.

Gibbs' Phenomenon



Gibbs phenomenon in square wave approximation



Overshoot behavior as K increase

The Fourier Transform

Definition (The Fourier Transform)

The Fourier Transform is an integral transform which takes a function as input, and outputs a function in a different domain.

$$\mathcal{F}\{g(t)\} = G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-2\pi i f t}\, dt$$

Remark

One could think about inputting $g(t)$ in the time domain, and recieving an output of $G(f)$, a function in the frequency domain.

Remark

With the Fourier Series showing us how to rewrite any periodic function as a sum of sinusoids, the Fourier Transform serves as an extension to non-periodic functions.

Fourier Transform Applications

Definition (Discrete Fourier Transform)

The Discrete Fourier Transform converts a finite sequence of time-domain samples into a finite sequence of frequency domain components. It captures the frequency content of a signal, and is used when a signal is finite and discrete. The DFT is extremely important for radio signals, sound waves, and even pixels in images. Along with that, it helps perform convolutions in PDEs, making it a useful tool in mathematics.

Fourier Transform Applications

Definition (Fast Fourier Transform)

The Fast Fourier Transform is an extremely efficient algorithim for calculating the Discrete Fourier Transform. The DFT takes $O(n^2)$ time to compute, as it takes n sums with n terms. Here, the O is in Big O notation, which allows us to describe how fast an algorithim grows as n increases. The FFT however, only takes $O(n \log n)$ time to compute, making the FFT much more practical in real usage. It has all of the properties of the DFT, but the increase in speed gives it a far greater advantage in the practical setting

Fourier Transform Applications

Definition (Quantum Fourier Transform)

The Quantum Fourier Transform (QFT) is a useful tool in Quantum Computing, as it is a linear transformation on quantum bits. It is the quantum analogue of DFT. It is extremely important in many different quantum algorithims, notably Shor's Algorithim (which is used for factoring). The QFT, like the FFT, has a notable advantage when it comes to speed, as it has a considerable speedup compared to the traditional DFT. This enables Quantum Computers to do cryptography, arithmetic, and signal analysis at an extremely fast rate.

Plancherel Theorem

Theorem (Plancherel Theorem)

If $f(x) \in L^2(\mathbb{R})$, and $\hat{f}(\xi)$ is its Fourier transform, then:

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi$$

Remark

If we denote energy as the squared magnitude of the function, integrated over the entire domain, Plancheral's theorem shows us that said energy is the same in both the time and frequency domain. This shows the Fourier Transform is a unitary operator on $L^2(\mathbb{R})$.

Characteristic Functions in Probability Theory

Definition

If we take a function f, with the properties of a Probability Density Function (PDF)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 1$$

Then the characteristic function of f is formed by the complex fourier coefficients, and the characteristic function is the Fourier Transform of the PDF.

Remark

This provides a useful alternative to dealing with PDFs, CDFs, and DFs in Probability Theory. Similar logic also applies if an r.v. has a moment-generating function, $M_X(x)$

Characteristic Functions in Probability Theory

Example. Let X be a random variable with density

$$f_X(x) = \frac{x^m}{m+1}, \quad x \in [0,1],$$

where $m \in \mathbb{N}$. The characteristic function is:

$$\varphi_X(t) = \int_0^1 e^{itx} \cdot \frac{x^m}{m+1}\, dx = \frac{m!\left(1 - e^{it} e_m(-it)\right)}{(-it)^{m+1}(m+1)},$$

where the function $e_n(x)$ is the **partial exponential function**:

$$e_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Adapted from: MATH 154, Harvard, Pg. 2

Conclusion

To conclude on our exciting journey through the world of Fourier Analysis, I would implore the reader to learn more about Fourier Series, the Fourier Transform, and all of the resulting studies. Although both the Fourier Series and Fourier Transform have a well defined role in Partial Differential Equations, they serve a much greater role in the larger field of mathematics. They are a core part of modern analysis, and have wide reaching impact, from number theory to the pricing of financial derivatives.