AN EXPOSITION TO FOURIER ANALYSIS

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Abstract. After the discovery of the Fourier Series by Fourier himself, the modeling of both periodic and non-periodic functions became an essential aspect of the ever-evolving field of Analysis. The goal of this paper is to understand the wide-reaching scope of Fourier Analysis, and go through many of the most studied ideas in the field.

1. Introduction

1.1. **Historical Background.** The Fourier Series and its resulting outreach can be traced back to many of the greatest mathematicians, including but not limited to Euler and Bernoulli. Of course, the series is named after, and was discovered by, French mathematician Jean-Baptiste Joseph Fourier in a search for solutions to the heat equation. With the Fourier Series itself being an expansion of trigonometric functions (explored in depth in this paper), it makes for a useful mathematical tool. The properties of trigonometric functions are well understood, making it easy (with ease being relative, we see later that the true power of Fourier Series lies in their orthogonality) to analyze functions as a sum of trigonometric functions (very similar to the commonly utilized Taylor series, which aims to accomplish the same goal in the form of polynomials.) Joseph Fourier himself noted that smooth, periodic functions have Fourier Series that converge to themselves, which makes the convergence of Fourier Series something that has been well studied, and well explored.

Past the periodic function, the Fourier transform has been explored similarly to model non-periodic functions, creating several notable results and theroems. Fourier Series has proved to be essential to the understanding of Differential Equations. Due to the Fourier Transform being an integral transform (something explored later on), several unique properties and identities emerge, with very notable uses in computing, information theory, and quantum mechanics.

1.2. **Aim.** This paper aims to provide an exposition into Fourier Analysis as a whole, with a strong focus on the properties and resutls of the Fourier Transform. We start by exploring the properties of the Fourier Series and their convergence. We then aim to explore the Fourier Transform in non-periodic functions, understanding the Uncertainty Theorem and basis of the Fourier Transform as an improper Riemann Integral. Following that, we look to explore results with practical implications, such as Plancheral's Theorem, Parseval's Identity, and the Poisson Summation Formula. Finally, we analyze characteristic functions in Probability Theory, and similar applications in Ergodic Theory.

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2. Preliminaries

Orthogonality. Two functions $f, g \in L^2([-\pi, \pi])$ are said to be orthogonal if

$$\langle f, g \rangle = 0.$$

Euler's Formula.

$$e^{ix} = \cos x + i \sin x.$$

Real Fourier Coefficients. Let $f \in L^2([-\pi, \pi])$. The real Fourier coefficients are defined by

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

Complex Fourier Coefficients.

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$$

Fourier Series Representations. The real Fourier series of f is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right].$$

The complex Fourier series is given by

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Parseval's Identity. If $f \in L^2([-\pi, \pi])$, then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2 \right),$$

or in the complex form,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Conditions for Convergence.

- **Dirichlet conditions:** These require that the function $f(x)$ be zero at the endpoints of the interval:
$$f(a) = f(b) = 0.$$
- **Neumann conditions:** These require that the derivative of the function vanish at the endpoints:
$$f'(a) = f'(b) = 0.$$
- **Periodic conditions:** These require that both the function and its derivatives match at the endpoints:
$$f(a) = f(b), \quad f'(a) = f'(b), \quad f''(a) = f''(b), \text{ etc.}$$

3. The Fourier Series

3.1. **Definition.** By definition, the Fourier Series is a way to define a periodic function as an infinite sum of sines and cosines, in order to make the function easier to understand. Given a periodic function $f(x)$, we can define it's Fourier Series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)] \tag{3.1}$$

Where a_0 is the cosine term evaluated at 0 (which is not required for the sine term, as the value of $sin(0)$ is 0. In this expression, a_n and b_n are the coefficients of sine and cosine respectively.

3.2. **Orthogonality.** One of the main goals while studying Fourier Series is to study our coefficients, of sin and cos respectively. To do so, we have to study the Orthogonality of functions. In a vector space, if the dot product of two vectors are zero, they are orthogonal. Similarly, if two functions, $f(x)$ and $h(x)$ are orthogonal if

$$\int_a^b g(x)\, h(x)\, dx = 0$$

Given the property that Fourier Series are indeed Orthogonal, we can use

Theorem 3.1 (Orthogonality of Trigonometric System)**.** The system

$$\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \ldots\}$$

is a complete orthogonal system from $[-\pi, \pi]$. To show this system is orthogonal, we must show that the following integrals vanish under our given conditions.

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\, dx &= 0, \qquad m, n \geq 0,\ m \neq n \\
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\, dx &= 0, \qquad m, n \geq 1,\ m \neq n \\
\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\, dx &= 0, \qquad m \geq 0,\ n \geq 1
\end{aligned}$$

3.2.1. *Computing Coefficients.*

Proof. By leveraging the integral properties and orthogonality of Fourier series, we can express a function $f(x)$ as a Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right].$$

To determine the coefficient a_n, multiply both sides by $\cos(mx)$ for $m \geq 1$, and integrate over the interval $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} f(x) \cos(mx)\, dx = \int_{-\pi}^{\pi} \left(a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right] \right) \cos(mx)\, dx.$$

We now apply the orthogonality identities:

$$\int_{-\pi}^{\pi} \cos(nx) \cos(mx)\, dx = \begin{cases} 0, & n \neq m \\ \pi, & n = m \end{cases}, \quad \int_{-\pi}^{\pi} \sin(nx) \cos(mx)\, dx = 0.$$

This implies that all terms in the expansion vanish except for the one where $n = m$. Therefore:

$$\int_{-\pi}^{\pi} f(x) \cos(mx)\, dx = a_m \int_{-\pi}^{\pi} \cos^2(mx)\, dx = a_m \pi.$$

Solving for a_m, we obtain:

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx)\, dx. \tag{3.2}$$

Similarly, to compute the coefficient b_m, we follow the same process but multiply by $\sin(mx)$ instead:

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx)\, dx. \tag{3.3}$$

For further details on the role of orthogonality in Fourier series, refer to [McM10]. ∎

3.2.2. *Example: Calculating Fourier Series and Fourier Coefficients.* Define the even square wave function $f(x)$ on the interval $[-\pi, \pi]$ as:

$$f(x) = \begin{cases} 1, & |x| < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq |x| \leq \pi \end{cases}$$

and extend it periodically with period 2π

Since $f(x)$ is even, its Fourier series contains only cosine terms (due to sine terms being odd, they are not represented in this Fourier Series):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

The coefficients are given by:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = 1$$

$$a_n = \frac{2}{\pi} \int_{0}^{\pi/2} \cos(nx)\, dx = \frac{2}{\pi n} \sin\left(\frac{n\pi}{2}\right)$$

So:

$$a_n = \begin{cases} \frac{2}{\pi n}(-1)^k, & n = 2k+1 \text{ (odd)} \\ 0, & n \text{ is even} \end{cases}$$

Thus, the Fourier series expansion of the even square wave is:

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\cos(x) - \frac{1}{3}\cos(3x) + \frac{1}{5}\cos(5x) - \cdots\right)$$

3.3. **Gibb's Phenomenon and Partial Sums.**

Theorem 3.2 (Partial Sums)**.** *A partial sum of the Fourier series of a given function f is shown in the where a function f is integrable on the segment $[-\pi, \pi]$, and*

$$S(f) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx))$$

is its Fourier series, the partial Fourier sum $S_n(f)$ of order n of f is the trigonometric polynomial

$$S_n(f) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx))$$

Theorem 3.3 (Gibb's Phenomenon)**.** *Take f, a piecewise, continously differentiable periodic function that contains a jump discontinuity. The Gibb's Phenomenon describes a behavior of the Nth partial Fourier series of f to create larger peaks around the jump, both overestimating and underestimating the function by nine percent with larger sine values. However, taking the infinite Fourier Series, the sum does converge everywhere.*

This is caused by the fact that functions of type f exhibit pointwise convergence, NOT uniform convergence (see 2). The Fourier Series of f converges to the function at every point other than that of the jump discontinuity. One way to combat issues that arise due to the Gibb's Phenomena is by using a smoother summation, such as the Fejér Kernel or Cesaro Summation.

3.4. **Convergence of Fourier Series.** The convergence of a Fourier series of a given periodic function $f(x)$ is a question that has been studied deeply in harmonic analysis. In this section, our aim is to study pointwise and uniform convergence.

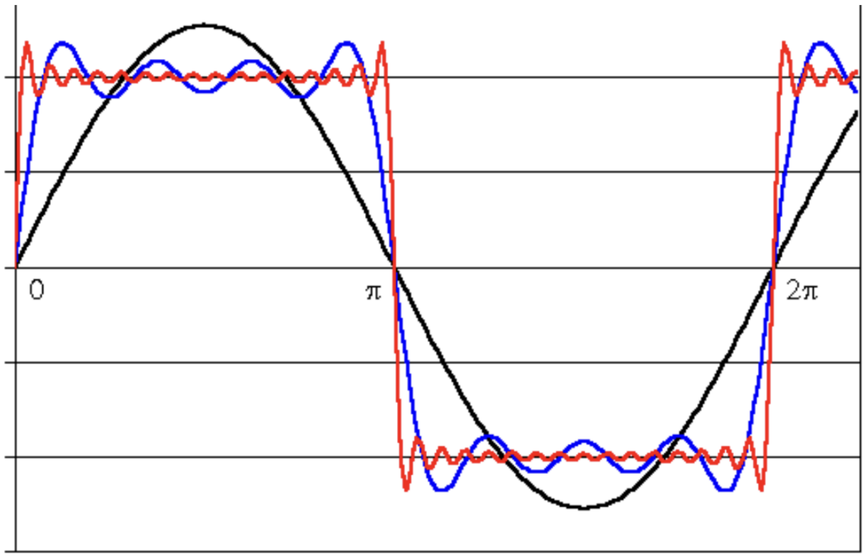

Figure 1. An example of the Fourier Series of the Square Wave, where the partial sums slowly converge to the actual function as more partial sums are taken (Adapted from Chemicool Dictionary)

3.4.1. *Pointwise Convergence.*

Definition 3.4 (Pointwise Convergence)**.** The mathematical definition of pointwise convergence is the idea that a sequence of functions can convergence to a very particular function on a point by point basis, not over the entire interval.

Theorem 3.5. *Let f be a piecewise continuous function on the interval $[-L, L]$, and let $\tilde{f}(x)$ be defined by*

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } f \text{ is continuous at } x \\ \frac{1}{2}\left(f(x^+) + f(x^-)\right), & \text{if } f \text{ is discontinuous at } x \end{cases}$$

Then the partial sums of the Fourier series $S_N(x)$ satisfy:

$$\lim_{N\to\infty} S_N(x) = \tilde{f}(x) \quad \text{for all } x \in (-L, L)$$

Notice how this does not satisfy on the endpoints, of $-L$ and L

Definition 3.6 (Pointwise Convergence of Fourier Series)**.** A Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges *pointwise* to a function $f(x)$ on an interval $[a, b]$ if for every $x \in [a, b]$,

$$\lim_{N\to\infty} S_N(f; x) = f(x),$$

where $S_N(f; x)$ is the N-th partial sum of the series.

At points where f is continuous, the series converges to $f(x)$. At jump discontinuities, it converges to the average of the left- and right-hand limits:

$$\lim_{N\to\infty} S_N(f; x_0) = \frac{1}{2}\left(f(x_0^-) + f(x_0^+)\right).$$

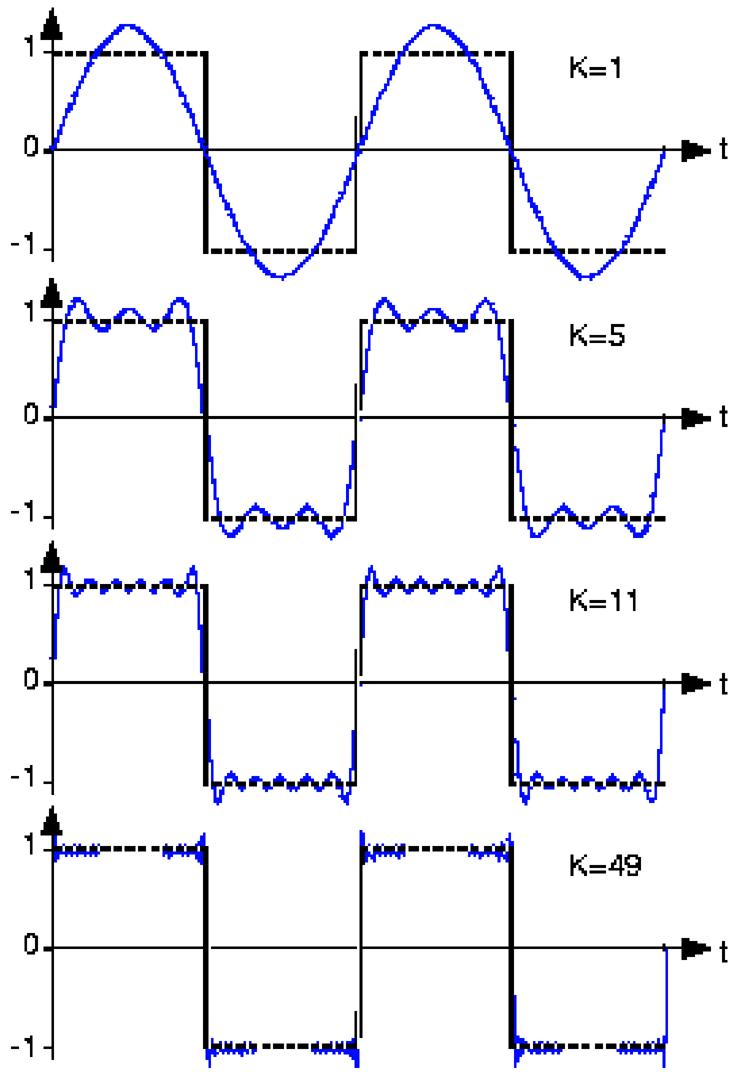

Figure 2. An example of the Gibbs Phenomena, where it can be noted that the overshooting trends to 9% as k increases

This result is a consequence of Dirichlet's theorem on Fourier series convergence. While pointwise convergence is useful, it does not preserve all properties of f, such as continuity or boundedness.

Theorem 3.7 (Dirilecht Kernel)**.** *The convolution of $D_n(x)$ of any function f with period 2π is the nth degree Fourier Series approximation to f. We have*

$$(D_n * f)(x) = \int_{-\pi}^{\pi} f(y)D_n(x-y)\,dy = 2\pi \sum_{k=-n}^{n} \hat{f}(k)e^{ikx},$$

where

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx$$

The motivation for the Dirilecht Kernel when it comes to pointwise convergence of Fourier Series is that when we expand out the Fourier Series with the expression for coefficients, we define the Dirilecht Kernel and its related series. By looking at its relation to the geometric series and resubstiting into the original cosine series, we find a direct integral for the partial sum of the Fourier Series. Pointwise convergence often fails at the endpoints, which introduces the need for stronger convergence.

3.4.2. *Uniform Convergence of Fourier Series.*

Definition 3.8 (Uniform Convergence)**.** A series of functions can be described as having uniform convergence if they converge to the desired functions at all points.

Theorem 3.9 (Uniform Convergence of Fourier Series)**.** *A Fourier series* $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ *converges* uniformly *to a function* $f(x)$ *on an interval* $[a, b]$ *if*

$$\lim_{N\to\infty} \sup_{x\in[a,b]} |S_N(f; x) - f(x)| = 0,$$

where $S_N(f; x)$ *denotes the* N*-th partial sum of the series.*

Uniform convergence is a stronger condition than pointwise convergence. It guarantees that the Fourier series approximates $f(x)$ with equal accuracy across the entire interval, not just at individual points. This implies that properties such as continuity and boundedness are preserved in the limit.

Uniform convergence is especially important in applications where global error bounds or numerical stability are necessary.

Theorem 3.10 (Rules of Uniform Convergence of Fourier Series)**.** *The Fourier series converges to* $f(x)$ *uniformly in* $[a, b]$, *if*

(i) $f(x)$ *is continuous, and* $f'(x)$ *is piecewise continuous on* $[a, b]$.
(ii) $f(x)$ *satisfies the associated boundary conditions.*

The classical Fourier series requires different boundary conditions depending on the form of the series. Dirichlet conditions for the sine series, Neumann conditions for the cosine series, and periodic conditions for the full Fourier series. See [Gri12] for a more in-depth discussion.

3.5. **Convolution Theorem.** One of the most practical results that come from the study of Fourier Series is the relationship between convolution in the time domain and multiplication in the frequency domain.

$$\mathcal{F}(f * g)(n) = \hat{f}(n) \cdot \hat{g}(n) \tag{3.2}$$

The convolution $f * g$ is defined on the interval $[-\pi, \pi]$ as

$$(f * g)(x) = \int_{-\pi}^{\pi} f(x - y)g(y)\, dy,$$

Assuming both f and g are integrable and have a period of 2π

This result reveals that a convolution blending or smoothing operation in the spatial domain corresponds to a simple multiplication of Fourier coefficients. To smooth a function, we can multiply its high-frequency coefficients by small values.

While a rigorous proof of the convolution theorem requires deeper analysis of convergence and periodic extensions, the idea that it can make operations that are complex in one domain, simple in another is the very property makes Fourier Series and the Fourier Transform essential in many mathematical fields, notably the study of Partial Differential Equations, where they become a powerful tool in the process of solving.

4. The Fourier Transform

4.1. **Fourier Transform.** In the last section, we thoroughly explored the properties of Fourier Series and how they're used to model periodic functions.

Definition 4.1 (Fourier Transform)**.** We can define the Fourier Transform of a function $g(t)$ as

$$\mathcal{F}\{g(t)\} = G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi ift}\,dt$$

In this case, we are left with a function of f, or frequency. $G(f)$ helps us understand how much power $g(t)$ has at a certain frequency f. This idea can be applied in many different ways.

Lets take the example of the square wave, in which we previously discussed it's decomposition as a sum of trignometric functions, or it's Fourier Series. We will soon extend this idea to non-periodic functions, but the square wave is a nice starting point.

Seeing as we used an even square wave when calculating Fourier Series, we can show the Fourier Transform with an odd square wave.

Proof. Define the time-domain function $g(t)$ as:

$$g(t) = \begin{cases} 1, & -a < t < 0 \\ -1, & 0 < t < a \\ 0, & \text{otherwise} \end{cases}$$

This is an odd, non-periodic square pulse of width $2a$, centered at the origin.

We compute the Fourier Transform of $g(t)$ with our previously defined definition:

$$\mathcal{F}\{g(t)\} = G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-2\pi ift}\,dt$$

Since $g(t)$ is only nonzero on from $-a$ to a, we split the integral:

$$\begin{aligned} G(f) &= \int_{-a}^{0} 1 \cdot e^{-2\pi ift}dt + \int_{0}^{a} (-1) \cdot e^{-2\pi ift}dt \\ &= \int_{-a}^{0} e^{-2\pi ift}dt - \int_{0}^{a} e^{-2\pi ift}dt \end{aligned}$$

Evaluate both integrals:

$$\begin{aligned} G(f) &= \left[\frac{e^{-2\pi ift}}{-2\pi if}\right]_{-a}^{0} - \left[\frac{e^{-2\pi ift}}{-2\pi if}\right]_{0}^{a} \\ &= \frac{1 - e^{2\pi ifa}}{2\pi if} - \frac{e^{-2\pi ifa} - 1}{2\pi if} \end{aligned}$$

$$= \frac{2 - \left(e^{2\pi i f a} + e^{-2\pi i f a}\right)}{2\pi i f} = \frac{2(1 - \cos(2\pi f a))}{2\pi i f}$$

$$G(f) = \frac{1 - \cos(2\pi f a)}{\pi i f}$$

This is the Fourier Transform of the odd square wave. It is purely imaginary and odd, being a reflection of the time-domain input. ■

4.2. **Inverse Transform.** Seeing as we just took the Fourier Transform of $g(t)$ in the time domain and received an output $G(f)$ on the frequency domain, one might ask how to go from the frequency domain to the time domain. In order to do so, we must utilize the Inverse Fourier Transform.

Theorem 4.2 (Inverse Fourier Transform)**.** *We can define the Inverse Fourier Transform as*

$$\mathcal{F}^{-1}\{G(f)\} = \int_{-\infty}^{\infty} G(f)\, e^{2\pi i f t}\, df = g(t).$$

This gives us a useful tool to take functions from the frequency domain and transform them into functions of the time domain.

Revisiting $G(f)$, our Fourier Transform of $g(t)$, an odd square wave, we can perform the Inverse Fourier Transform to get $g(t)$

$$G(f) = \frac{1 - \cos(2\pi f a)}{\pi i f}$$

We compute the inverse Fourier transform with:

$$g(t) = \mathcal{F}^{-1}\{G(f)\} = \int_{-\infty}^{\infty} G(f) e^{2\pi i f t}\, df$$

Substitute $G(f)$ into the integral:

$$g(t) = \int_{-\infty}^{\infty} \frac{1 - \cos(2\pi f a)}{\pi i f}\, e^{2\pi i f t}\, df$$

Split the integral using linearity:

$$g(t) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{2\pi i f t}}{f}\, df - \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\cos(2\pi f a)}{f} \cdot e^{2\pi i f t}\, df$$

Now write $\cos(2\pi f a)$ in exponential form:

$$\cos(2\pi f a) = \frac{e^{2\pi i f a} + e^{-2\pi i f a}}{2}$$

Substitute into the second integral:

$$g(t) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{2\pi i f t}}{f}\, df - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{e^{2\pi i f (t+a)} + e^{2\pi i f (t-a)}}{f} \right) df$$

Group terms:

$$g(t) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{e^{2\pi ift}}{f}\, df - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{2\pi if(t+a)}}{f}\, df - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{2\pi if(t-a)}}{f}\, df$$

Let us define each of these integrals symbolically:

$$I_1 = \int_{-\infty}^{\infty} \frac{e^{2\pi ift}}{f}\, df, \quad I_2 = \int_{-\infty}^{\infty} \frac{e^{2\pi if(t+a)}}{f}\, df, \quad I_3 = \int_{-\infty}^{\infty} \frac{e^{2\pi if(t-a)}}{f}\, df$$

Thus, we have:

$$g(t) = \frac{1}{\pi i} I_1 - \frac{1}{2\pi i}(I_2 + I_3)$$

These integrals do not converge classically. We can denote these as Cauchy Principal Value Integrals. Our combination of integrals gives us our original time domain function. Refer to [SW71] for a more indepth discussion.

$$g(t) = \begin{cases} 1, & -a < t < 0 \\ -1, & 0 < t < a \\ 0, & \text{else} \end{cases}$$

This confirms that the inverse Fourier transform of $G(f)$ yields the original odd square wave.

4.3. **General Properties.** In this section, we will go over many of the essential properties of the Fourier Transform that allow it to be used so generally.

(1) Linearity Property

Definition 4.3 (Linearity of Fourier The Transform)**.** Take any two functions in the time domain, $g(t)$ and $f(t)$. Say that both functions have Fourier Transforms $G(f)$ and $F(f)$ respectively. The Fourier Transform of any linear combinations of our time domain functions can be expressed as

$$\mathcal{F}\{c_1 g(t) + c_2 f(t)\} = c_1 G(f) + c_2 F(f) \tag{4.1}$$

(2) Shifts Property

Definition 4.4 (Shifts Property of the Fourier Transform)**.** Take a function $g(t)$ in the time domain, with a Fourier Transform $G(F)$. The Fourier Transform of $g(t-a)$ where a is a real number can be denoted as

$$e^{-i2\pi fa}\, G(f)$$

Which is shown by

$$\begin{aligned} \mathcal{F}\{g(t-a)\} &= \int_{-\infty}^{\infty} g(t-a)\, e^{-i2\pi ft}\, dt \\ &= \int_{-\infty}^{\infty} g(u)\, e^{-i2\pi f(u+a)}\, du \\ &= e^{-i2\pi fa} \int_{-\infty}^{\infty} g(u)\, e^{-i2\pi fu}\, du \\ &= e^{-i2\pi fa} G(f) \end{aligned}$$

(3) Scaling Property

Definition 4.5 (Scaling Property of the Fourier Transform)**.** Take a function $g(t)$ in the time domain and scalar $c \in \mathbb{R}$. The Fourier Transform of $g(ct)$ can be represented as

$$\mathcal{F}\{g(ct)\} = \frac{1}{|c|}\, G\left(\frac{f}{c}\right)$$

Proof.

$$\mathcal{F}\{g(ct)\} = \int_{-\infty}^{\infty} g(ct)\, e^{-i2\pi ft}\, dt$$

Substitute: $u = ct, \quad du = c\, dt$

$$\mathcal{F}\{g(ct)\} = \int_{-\infty}^{\infty} \frac{g(u)}{c}\, e^{-i2\pi f\frac{u}{c}}\, du$$

Where once we integrate under the dual pretense that c is either negative or positive, we receive our Fourier Transform for $g(ct)$

$$\mathcal{F}\{g(ct)\} = \frac{1}{|c|}\, G\left(\frac{f}{c}\right)$$

∎

(4) Derivative Property

Definition 4.6 (Derivative Property of the Fourier Transform)**.** Take a function $g(t)$ in the time domain. The Fourier Transform of the derivative of $g(t)$ can be shown as

$$\mathcal{F}\left\{\frac{dg(t)}{dt}\right\} = i2\pi f \cdot G(f)$$

(5) Modulation Property

Definition 4.7 (Modulation Property of the Fourier Transform)**.** Take two functions, $g(t)$ and $f(t)$ in the time domain. We can say that these two functions are modulated if they are multiplied in the time domain. We can take the Fourier Transform of their modulation with

$$\mathcal{F}\{g(t)h(t)\} = G(f) * F(f)$$

Where $G(f)$ and $F(f)$ are the Fourier Transforms of $g(t)$ and $f(t)$ respectively.

4.4. **Plancherel Theorem.**

4.4.1. *Plancherel Theorem.* Plancherel's Theorem is an important result in modern analysis, which showcases the unitary of the Fourier Transform.

Definition 4.8 (Unitary Transformation)**.** A Unitary Transformation is a linear isomorphism that preserves the inner product. The inner product of two functions before the transformation is equivalent to the inner product after the transformation.

Theorem 4.9 (Plancherel Theorem)**.**

$$\int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} |G(f)|^2\, df \tag{4.2}$$

In layman's terms, this lets us know that the energy over the Fourier Transform is preserved (where the integral of the squared magnitude represents energy). Taking the integral of the squared modulus over the interval of all $\mathbb{R}$ gives us the crucial identity that the Fourier Transform is unitary.

This theorem has also been known as Plancherel-Parseval's Theorem, but the general idea of the Fourier Transform being unitary is preserved, regardless of notation.

Remark 4.10. What this also gives us is the notion that, if $g(t)$ belongs to L^1 and L^2, $G(f)$ also belongs to L^2, which can be shown as a reiteration of the original theorem.

4.5. **Poisson Summation Formula.**

Definition 4.11 (Schwartz Function)**.** We can define any function f as a Schwartz Function if f is smooth, and is of rapid decay

$$|f(x)| \ll |x|^{-N} \quad \text{as } x \to \infty \text{ for all } N.$$

In simpler terms, where the derivative of f is rapidly decreasing

Remark 4.12. The Fourier Transform of a Schwartz Function is also a Schwartz Function

Theorem 4.13 (Poisson Summation Formula)**.** *Let $g(n)$ be a function in the time domain, and $G(n)$ be its Fourier Transform. The Poisson Summation Formula states*

$$\sum_{n \in \mathbb{Z}} g(n) = \sum_{n \in \mathbb{Z}} G(n)$$

Proof. Define

$$G(x) = \sum_{n \in \mathbb{Z}} g(x+n).$$

Since the sum converges absolutely, $G(x)$ is 1-periodic and integrable on $[0,1]$, so it has a Fourier series. Its Fourier coefficients are:

$$\widehat{G}(k) = \int_0^1 \sum_{n \in \mathbb{Z}} g(x+n) e^{-2\pi i k x} \, dx.$$

By absolute convergence and the fact that g is a Schwartz function (rapidly decreasing and smooth), we can swap summation and integration:

$$\widehat{G}(k) = \sum_{n \in \mathbb{Z}} \int_0^1 g(x+n) e^{-2\pi i k x} \, dx.$$

$u = x + n$ yields:

$$= \sum_{n \in \mathbb{Z}} \int_n^{n+1} g(u) e^{-2\pi i k (u-n)} \, du = \sum_{n \in \mathbb{Z}} \int_n^{n+1} g(u) e^{-2\pi i k u} \, du.$$

We see this is a sum of integrals over disjoint intervals covering all of $\mathbb{R}$, so we get:

$$\widehat{G}(k) = \int_{\mathbb{R}} g(u) e^{-2\pi i k u} \, du = \widehat{g}(k).$$

So the Fourier series of G is:

$$G(x) = \sum_{k \in \mathbb{Z}} \widehat{g}(k) e^{2\pi i k x}.$$

Evaluating at $x = 0$, we conclude:

$$\sum_{n\in\mathbb{Z}} g(n) = G(0) = \sum_{k\in\mathbb{Z}} \widehat{g}(k),$$

∎

5. Applications of Fourier Analysis

5.1. Characteristic Function in Probability Theory.

Definition 5.1 (Probability Density Function)**.** In Probability Theory, we can say that a non-negative function f which has the property

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 1$$

can be called a Probability Density Function (PDF).

Definition 5.2 (Characteristic Function)**.** We can say that the complex Fourier Coefficients of f,

$$c_n = \mathbb{E}\left[fe^{inx}\right]$$

are called the characteristic function of the distribution.

Theorem 5.3. *What this gives us is a unique identity, if g and f are distribution functions of independent data, this convolution*

$$f \star g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)\,dy$$

represents the distribution of the sum of the data from g and f. An indepth discussion and proof can be found on [Kni25]

Theorem 5.4. *Let $\{X_k\}_{k=1}^{m}$ be a sequence of independent random variables on the circle, each with a density function possessing Fourier coefficients $\{c_n\}$. Then, as $m \to \infty$, the distribution of the normalized sum of the X_k converges to the uniform (constant) distribution on the circle.*

Proof. Suppose each density function f_k has Fourier coefficients $\{c_n\}$. Because the data are independent, the Fourier coefficients of the distribution of the sum are given by c_n^m. Since $|c_n| < 1$ for all $n \neq 0$, we have $c_n^m \to 0$ as $m \to \infty$.

In the limit, the only surviving Fourier coefficient is c_0, which corresponds to the mean (and equals 1 for normalized densities). Thus, the limiting distribution has all higher frequencies vanishing and only the constant term remaining. This is the uniform distribution on the circle. ∎

5.2. Discrete Fourier Transform.

Definition 5.5 (Discrete Fourier Transform)**.** The Discrete Fourier Transform (DFT) converts equally spaced samples of a function into a sequence with the same length, but samples of the Discrete Time Fourier Transform (DTFT). The DTFT is a complex valued function of frequency

Definition 5.6 (Discrete Time Fourier Transform)**.** The Discrete Time Fourier Transform (DTFT) is a method in Fourier Analysis used to analyze samples of a continous function. It produces a function of frequency that is a periodic summation of the continuous Fourier Transform. The DTFT is a continuous function of frequency, but in application we often take discrete samples via the DFT.

$$G(f) \triangleq \sum_{k=-\infty}^{\infty} \Delta t \cdot g(k\Delta t)\, e^{-i2\pi f k \Delta t}$$

- $G(f)$ is the Fourier transform of the sampled signal.
- $g(k\Delta t)$ is the original continuous-time signal sampled at intervals of Δt.
- Δt is the sampling interval
- f is the continuous frequency variable (in Hz).
- k indexes the sampled points in time.

There are alternative definitions to the DTFT, but we'll use this definition henceforth for simplicity purposes.

The DFT has many uses in Fourier Analysis, including but not limited to

- Digital Signal Processing
- Image Processing
- Solving Partial Differential Equations

Theorem 5.7. *We can mathematically define the DFT as*

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi \frac{k}{N} n}$$

- X_k is the k-th frequency component.
- x_n is the n-th sample of the time-domain signal.
- N is the total number of samples.
- k is the index of the output frequency bin ($k = 0, 1, \ldots, N-1$).
- $e^{-i2\pi \frac{k}{N} n}$ is the complex exponential basis function.

5.2.1. *Fast Fourier Transform.* The Fast Fourier Transform is an efficient computer algorithm that computes the DFT. The FFT rapidly computes the DFT by factorizing the matrix produced when taking the DFT into a product of mostly zero factors.

Definition 5.8 (Big O Notation)**.** Big O notation is a mathematical notation that describes the behavior of a function when trending to its limit, whether that be at infinity or a constant N. We denote big O notation as

$$O(g(x))$$

In computer science, the field we are interested in when discussing the rapid speeds of calculation of the FFT, Big O notation is a useful tool to classify algorithms by their runtime as the input grows.

If we denote the DFT as taking $O(n^2)$ operations n sums with n terms, the FFT can be denoted as taking $O(n \log n$ operations. This is a significant saving, allowing for the FFT to be employed in far more efficient settings, and making it far more practical in actual use.

Definition 5.9 (Cooley-Turkey Algorithm)**.** This is the most commonly used FFT with the most widespread applications in computing. This recursively breaks down DFTs of size $n = n_1 n_2$ into smaller DFTs of size n_2. This also has a unique ability, where it can be combined with other DFT algorithms, making it essential in computations.

Other notable FFT algorithms include the Prime-factor FFT algorithm, Rader's FFT algorithm, and Hexagonal FFT.

5.2.2. *Quantum Fourier Transform.* The Quantum Fourier Transform is the Quantum analouge of the DFT. As we discussed earlier, the QFT has a significant runtime advantage compared to the DFT, making it essential in practice. In Quantum Computing, it is the linear transformation of Quantum bits. It is a part of many algorithms, notably Shor's Algorithm for factoring and computing the discrete logarithm, and plays a crucial role in many mathematical fields. It also enables the ability of Quantum Computers to do extremely fast arithmetic calculations.

Definition 5.10 (Quantum Fourier Transform)**.** We can define the QFT mathematically:
Consider a complex-valued vector

$$(x_0, x_1, \ldots, x_{N-1}) \in \mathbb{C}^N.$$

The classical discrete Fourier transform (DFT) maps this to another vector

$$(y_0, y_1, \ldots, y_{N-1}) \in \mathbb{C}^N$$

using the formula

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \, \omega_N^{-jk}, \quad \text{for } k = 0, 1, \ldots, N-1,$$

where $\omega_N = e^{2\pi i/N}$ denotes a primitive N-th root of unity.

In the quantum context, the QFT transforms a basis state $|x\rangle$ of the form

$$|x\rangle = \sum_{j=0}^{N-1} x_j \, |j\rangle$$

into a new state

$$|\tilde{x}\rangle = \sum_{k=0}^{N-1} y_k \, |k\rangle \,,$$

where the new amplitudes y_k are given by

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \, \omega_N^{jk}, \quad \text{for } k = 0, 1, \ldots, N-1.$$

6. Conclusion

Seeing as this paper has explored several different results in Fourier Series, both highly theoretical and highly practical, several questions emerge. One could explore the limitations of Fourier Series when applied to functions with highly localized discontinuities, or explore how the role of the QFT in the ever-evolving field of Quantum Computing could push the bounds of the practical applications of Fourier Analysis. It is heavily recommended that the reader study the wide-reaching impact of Fourier Analysis, all the way from undiscussed kernels to the applications of Fourier Analysis in fields like Number Theory or Ergodic Theory.

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