Too Close for Comfort: Rational Approximations and the Limits of Irrationality

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Introduction

- How closely can irrational numbers be approximated by rationals?
- Are certain irrational numbers better approximable than others?

Rational Approximations & Dirichlet's Theorem

- Every real number is arbitrarily approximable by rationals.
- For any real α and $n > 1$ $n \geq q$, there exist integers p, q such that
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{nq}$$
- In particular, since $q \leq n$, we have
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Now, consider an integer $n > 1$. For every $k = 0, 1, \ldots, n$ we can write $k\alpha = x_k + y_k$ where x_k is an integer, and $0 \leq y_k < 1$. We can divide the interval of y_k, $[0, 1)$ into n smaller intervals of measure $\frac{1}{n}$ each.
We now have $n + 1$ numbers $y_0, y_1, \ldots, y_n$ but only n intervals. Therefore, by PHP, at least 2 of the numbers lie in the same interval. Let us say those numbers are y_i, y_j and assume $i > j$ without loss of generality.

Now:

$|(i - j)\alpha - (x_i - x_j)| = |(i\alpha - x_i) - (j\alpha - x_j)| = |y_i - y_j| < \frac{1}{n}$

Dividing both sides by $i - j$, we get

$$|\alpha - \frac{x_i - x_j}{i - j}| < \frac{1}{n \cdot (i - j)}$$

As required.

Improving the Bound: Hurwitz's Theorem

- For any irrational α, there are infinitely many $\frac{p}{q}$ with
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}\, q^2}.$$
- The constant $\sqrt{5}$ is best possible for all irrationals.

Lagrange Spectrum

- Define the bound

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Lq^2}.$$

  The Lagrange Spectrum $\mathcal{L}$ is the set of all L such that the above inequality is true for infinitely many rational $\frac{p}{q}$

- The smallest element of $\mathcal{L}$ is $\sqrt{5}$, corresponding to the golden ratio,

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

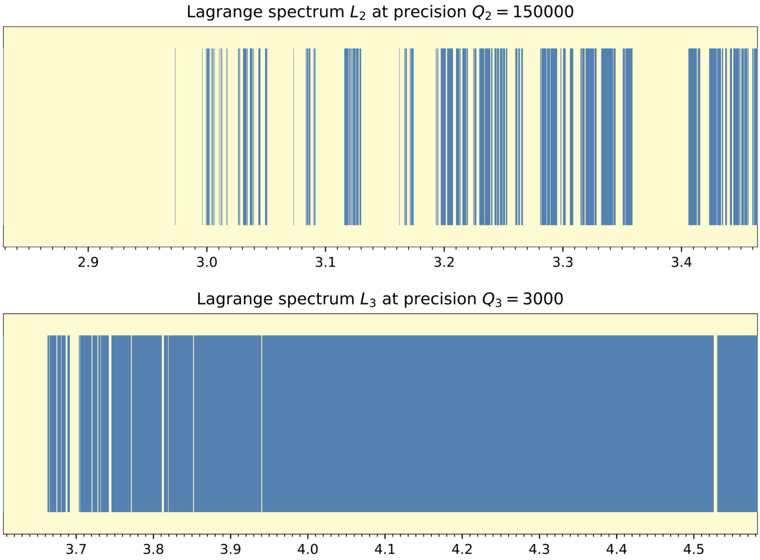

Figure: Parts of the Lagrange Spectrum visualised

Continued Fractions

- A simple continued fraction is a continued fraction of the form
$$\alpha = b_0 + \cfrac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 \cdots}}}$$
This can also be represented in the form $\alpha = [b_0; b_1, b_2, b_3, \ldots]$. Every real number has a simple continued fraction expansion.
- $b_i > 0 \; \forall \; i > 0$
- Truncating the fraction at the b_n gives the convergent $\frac{p_n}{q_n}$ with error bound
$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n \, q_{n+1}} \leq \frac{1}{q_n^2}.$$
This follows from the fact that all the denominators $q_n > 0$ fo $n > 1$ and Dirichlet's theorem.

Error of Approximation

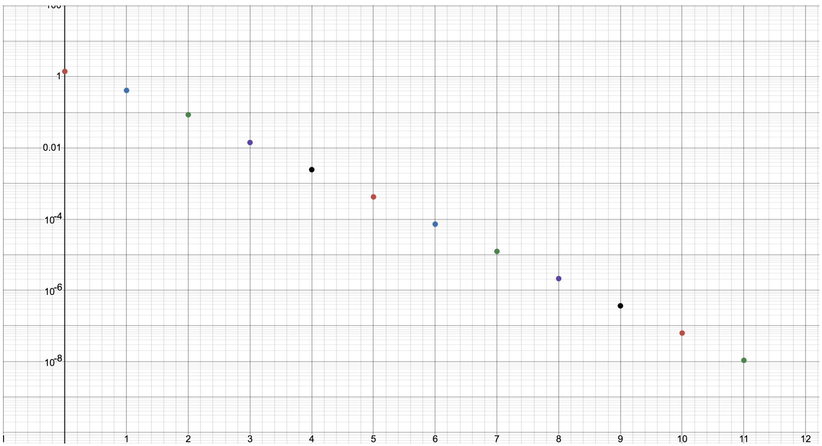

Figure: Error of approximation for the convergents of $\sqrt{2}$

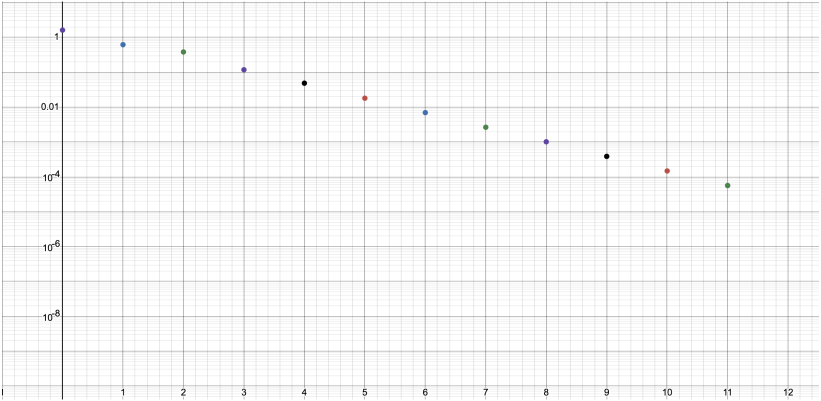

Figure: Error of approximation for the convergents of φ

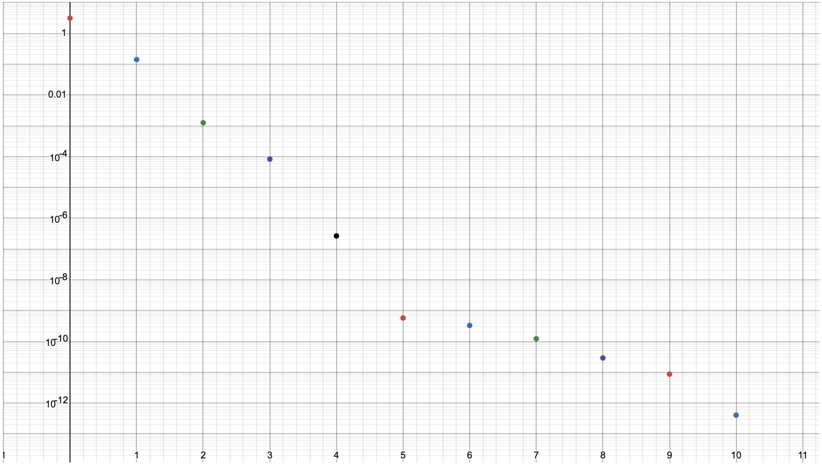

Figure: Error of approximation for the convergents of π

Quadratic Irrationals

- Quadratic irrationals are of the form $\frac{a+b\sqrt{n}}{c}$.
- They have eventually periodic continued fraction expansions.
- Example:

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2+\frac{1}{2\dots}}} = [1; \overline{2}].$$

- They are the only fractions with eventually periodic continued fractions.

Irrationality Exponent

- Define $\mu(\alpha)$ as the supremum of μ such that

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{\mu}}$$

holds for infinitely many $\frac{p}{q}$.

- For algebraic irrationals, $\mu(\alpha) = 2$.
- For Transcendental numbers, $\mu(\alpha) \geq 2$
- Certain numbers have irrationality exponent ∞. One such number is the Liouville number,

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{10^{k!}} = 0.110001000000000000000001\ldots$$

Markov Constant

- Define the Markov constant $M(\alpha)$ as the largest M such that
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Mq^2}$$
for infinitely many $\frac{p}{q}$.
- For φ, the Markov constant is
$$M(\varphi) = \sqrt{5}.$$