Too Close for Comfort: Diophantine Approximation and the Limits of Irrationality

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1 Introduction

How closely can irrational numbers be approximated by rational ones? Diophantine approximation seeks to answer this, blending number theory, geometry, and analysis to study the limits of rational approximation. Beginning with Dirichlet's foundational theorem and its refinement by Hurwitz, we trace how well irrationals can be approximated and how such bounds vary across numbers. Continued fractions emerge as a powerful tool, not only for constructing optimal approximations but also for revealing deep structure, such as the periodicity of quadratic irrationals.

From there, we explore measures of irrationality, like the irrationality exponent and the Markov constant, which quantify just how "resistant" a number is to approximation. Along the way, we meet Liouville numbers, the golden ratio, and the Lagrange spectrum, each offering a unique perspective on the rational–irrational divide.

This paper charts a path through these ideas, revealing a rich theory where geometry and arithmetic meet to expose the hidden precision in irrationality.

2 Rational Approximations and Dirichlet's Theorem

We begin with the following fact: every real number can be arbitrarily approximated by a rational number. That is, for any real α and any real $\epsilon > 0$, there exists a rational number $p/q \in \mathbb{Q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \epsilon$$

This follows from the fact that the rational numbers are dense in the reals.

But this only tells us that *some* approximation exists, not how we can approximate it with rationals. It does not tell us how large we need to make our denominator to get a certain level of accuracy. This motivates us to ask the following question. For a fixed q, how well can we approximate α to a real number p/q?

This leads naturally to a foundational result in Diophantine approximation: Dirichlet's Approximation Theorem.

Dirichlet's Approximation Theorem

For any real numbers α and an integer $n > 1$, there exists integers p and q such that $1 \leq q \leq n$ and

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{nq}$$

Proof:

Case 1: α is rational

In this case, simply take $\frac{p}{q} = \alpha$, so that $|\alpha|\frac{p}{q}| = 0$. Since $n > 0$ and $q > 0$, $nq > 0$. Hence, we have $0 < \frac{1}{nq}$, and hence $|\alpha - \frac{p}{q}| < \frac{1}{nq}$.

Case 2: α is irrational

For this case, we start by stating the following principle.

Pigeonhole Principle : if n items are put into m containers, with $n > m$, then at least one container must contain more than one item.

Now, consider an integer $n \geq 2$. For every $k = 0, 1, \ldots, n$ we can write $k\alpha = x_k + y_k$ where x_k is an integer, and $0 \leq y_k < 1$. We can divide the interval $[0, 1)$ into n smaller intervals of measure $1/n$ each. We now have $n + 1$ numbers $y_0, y_1, \ldots, y_n$ but only n intervals. Therefore, by the Pigeonhole Principle, at least 2 of the numbers lie in the same interval. Let us say those numbers are y_i, y_j and assume $i > j$ without loss of generality.

Now:

$$|(i-j)\alpha - (x_i - x_j)| = |(i\alpha - x_i) - (j\alpha - x_j)| = |y_i - y_j| < \frac{1}{n}$$

Dividing both sides by $i - j$, we get

$$|\alpha - \tfrac{x_i - x_j}{i-j}| < \tfrac{1}{n \cdot (i-j)}$$

As required.

(Q.E.D)

What this theorem really says is that real numbers, no matter how irrational, are never too far from rational ones. For any bound n, we can find a rational number p/q, with $q \leq n$, such that the error $|\alpha - p/q|$ is less than $1/nq$. Since $q \leq n$, this implies that

$$|\alpha - \frac{p}{q}| < \frac{1}{nq} \leq \frac{1}{q^2}.$$

In other words, we can always find rational approximations whose error is less than the square of the reciprocal of the denominator. Even the most irrational numbers are still, in a sense, too close to comfort to rational ones.

This now motivates us to ask: Can the $1/q^2$ bound be improved on? If so, how much can we improve it? If not, why not?

3 Improving Dirichlet: Hurwitz's Theorem

The bound given by Dirichlet's Theorem is already quite powerful. But surprisingly, it can be improved. If we restrict our attention to irrational numbers, then there is a stronger result — one that improves the approximation bound from $1/q^2$ to a higher constant multiple of it.

Hurwitz's Theorem

For any irrational number α, there exist infinitely many rational numbers $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

Hurwitz's Theorem improves Dirichlet's bound: instead of merely guaranteeing the existence of approximations satisfying $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$, it ensures that the error can be made even smaller — less than $1/\sqrt{5}q^2$ — and still occur infinitely often.

Note that no larger constraint works for *all* irrationals. That is, the constant $\sqrt{5}$ is the best possible bound that holds for all irrationals. This is because if we plug in $\alpha = \varphi$, there are only *finitely* many solutions to the inequality.

3.1 Farey Sequences

Let us start by defining a completely reduced fraction to be a fraction whose numerator and denominator are coprime. That is, a fraction $\frac{p}{q}$ is completely reduced if and only if $gcd(p, q) = 1$.

A Farey sequence of order n, denoted by $\mathcal{F}_n$, is defined to be the sequence formed by the set of all completely reduced fractions between 0 and 1 whose denominator does not exceed n, arranged in ascending order. Below are the first 6 Farey Sequences.

$$\mathcal{F}_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$
$$\mathcal{F}_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$$
$$\mathcal{F}_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$
$$\mathcal{F}_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$$
$$\mathcal{F}_5 = \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}$$
$$\mathcal{F}_6 = \left\{\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}\right\}$$

We now define the following operation on the rational numbers. We will call this the mediant.

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$$

A property of the mediant is that the mediant of any 2 rational numbers always lies strictly between them. That is, if $a/b < c/d$ then, $a/b < a/b \oplus c/d < c/d$

Proof: Assume $\frac{a}{b} < \frac{c}{d}$.

$$\begin{aligned}
\frac{a}{b} &< \frac{a+c}{b+d} : \\
\Leftrightarrow a(b+d) &< b(a+c) \\
\Leftrightarrow ab + ad &< ba + bc \\
\Leftrightarrow ad &< bc \\
\Leftrightarrow \frac{a}{b} &< \frac{c}{d}
\end{aligned}$$

$$\begin{aligned}
\frac{a+c}{b+d} &< \frac{c}{d} : \\
\Leftrightarrow d(a+c) &< c(b+d) \\
\Leftrightarrow da + dc &< cb + cd \\
\Leftrightarrow da &< cb \\
\Leftrightarrow \frac{a}{b} &< \frac{c}{d}
\end{aligned}$$

$$\therefore \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

Farey Term Generation via Mediants

If a/b, x/y and c/d are members of a Farey Sequence, and $a/b < x/y < c/d$,

$$\exists\, p, q \in \mathbb{Z}^{+}\; s.t\; \frac{x}{y} = \frac{ap}{bp} \oplus \frac{qc}{qd} = \frac{ap+qc}{bp+qd}$$

Proof:

Let $p = cy - dx$ and $q = bx - ay$. Both p, q must be positive integers since $c/d > x/y$ and $x/y > a/b$.

$$\begin{aligned} ap + qc &= a(cy - dx) + (bx - ay)c \\ &= acy - adx + cbx - acy \\ &= cbx - adx \\ &= (cb - ad)x \\ bp + qd &= b(cy - dx) + (bx - ay)d \\ &= bcy - bdx + bdx - ady \\ &= bcy - ady \\ &= (cb - ad)y \end{aligned}$$

$$\therefore \frac{ap+qc}{cp+qd} = \frac{(cb-ad)x}{(cb-ad)y} = \frac{x}{y}$$

3.1.1 Farey Neighbours

Two rational numbers are said to be Farey Neighbours if the appear next to each other in a Farey Sequence.

Property 1 of Farey Neighbours

If a/b and c/d are Farey Neighbours, $(a+b)/(c+d)$ is a completely reduced fraction.

Proof: Let us assume that there exists two rational numbers a/b and c/d that are neighbours in $\mathcal{F}_n$ such that $gcd(a+c, b+d) = g \geq 2$. Firstly, notice that $n \geq max(b, d)$. Secondly, notice that n is strictly less than $b+d$ as otherwise, $(a+c)/(b+d)$ lies between a/c and b/d which means they can't be neighbours.

This means that we can write

$$\frac{a+c}{b+d} = \frac{\frac{a+b}{g}}{\frac{b+d}{g}} = \frac{e}{f},\; gcd(e, f) = 1$$

Recall that $n \geq max(b, d)$. This means that $2n \geq b + d$. That is,

$$f = \frac{b+d}{g} \leq \frac{b+d}{2} \leq n$$

This means that e/f is a completely reduced fraction with denominator at most n. That is, it is in $\mathcal{F}_n$ — contradicting the assumption that a/b and c/d are rational numbers in $\mathcal{F}_n$.

(Q.E.D)

Property 2 of Farey Neighbours

a/b and c/d are Farey Neighbours *if and only if* $|bc - ad| = 1$

Proof:

We first prove that if $|bc-ad| = 1$, a/b and c/d are Farey Neighbours. Assume $a/b < c/d$ without loss of generality.

Now, assume a/b and c/d are not neighbours. Firstly, notice that if $\frac{a}{b} < \frac{c}{d}$,

$$\forall p, q \in \mathbb{Z}+, \frac{a}{b} < \frac{ap+c}{bp+d} < \frac{ap+qc}{bp+qd} < \frac{c}{d}$$

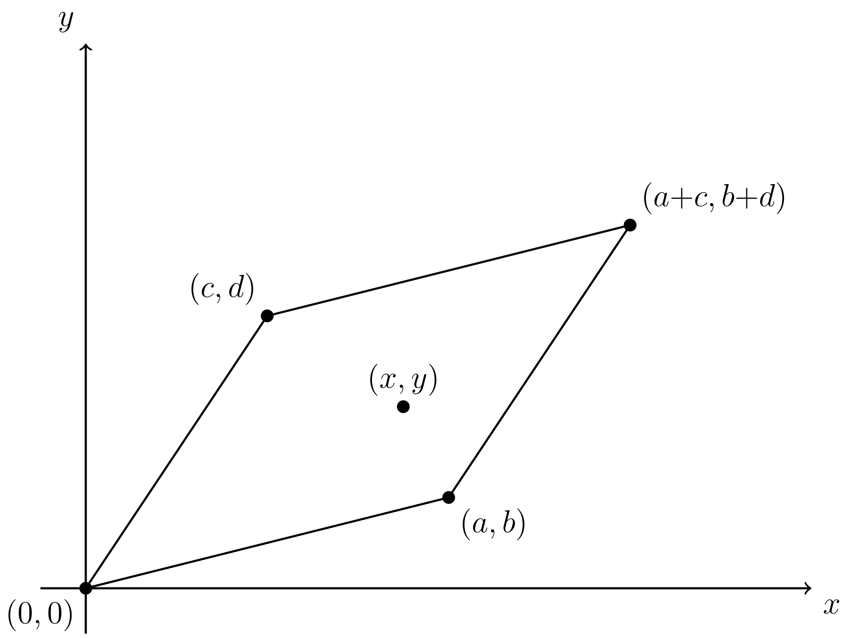
This is because

$$\frac{ap+c}{bp+d} - \frac{ap+qc}{bp+qd} = \frac{p(1-q)(bc-ad)}{(bp+qd)(bp+d)} < 0$$

Secondly, recall that every term between a/b and c/d is expressable in the form $\frac{ap+qc}{bp+qd}$ where $p, q \in \mathbb{Z}+$. Hence, the neighbour of a/b must be of the form $\frac{ap+c}{bp+d}$. Notice that $(ap+c)b-(bp+d)a = abp+bc-abp-ad = bc-ad = 1$. Hence, $gcd(ap+c, bp+d) = 1$. Hence, the fraction is irreducible.

This tells us that if $bc - ad = 1$, the denominator of the neighbour of a/b, if not b/d is of the form $bp + d \geq b + d \, \forall p \in \mathbb{Z}+$. Hence, for all n such that $max(b, d) \leq n < b + d$, a/b and c/d must be neighbours.

We will prove that $a/b, c/d$ are Farey Neighbours $\Rightarrow |bc - ad| = 1$ using a geometrical approach. Let a/b and c/d be Farey Neighbours in $\mathcal{F}_{b+d-1}$. We know that if a/b and c/d are neighbours in $\mathcal{F}_{b+d-1}$, they will also be neighbours in all $\mathcal{F}_n$ such that $max(b, d) \leq n < b + d$.



Consider the lattice points $(a, b), (c, d)$ and $(a+b, c+d)$. These three points, along with the origin $(0, 0)$ lie form the vertices of a parallelogram. Assume that this parallelogram has at least one interior lattice point, say, (x, y). By definition, $y < b + d$ and $a/b < x/y < c/d$. Hence, if such a point exists, it lies between a/b and c/d — a contradiction. Therefore, this parallelogram has no interior lattice point.

Now, we prove that this parallelogram has exactly four lattice points on its boundary, those being the vertices themselves.

Edge joining $(0, 0)$ and (a, b): This is represented by the line $y = \frac{b}{a}x, 0 \leq x \leq a$. By definition of a Farey Sequence, $gcd(a, b) = 1$. Hence, if $\frac{b}{a}x$ is an integer, $a|x$. Only such x in the relevant range are $x = 0, a$ which represent the vertices $(0, 0)$ and (a, b) respectively. Similarly, no lattice points except the vertices lie on the edge joining $(0, 0)$ and (b, a).

Edge joining (a, b) and $(a+b, c+d)$: This is represented by the line $y = \frac{d}{c}x + b - \frac{da}{c}, c \leq x \leq a+c$. If $\frac{d}{c}x + b - \frac{da}{c}$ is an integer, $\frac{dx - da}{c}$ is also an integer, which means $c|d(x-a)$. By definition of a Farey Sequence, $gcd(c, d) = 1$. Hence, $c|(x - a)$. Only such x in the relevant range are $x = a, a + c$ which represents the vertices (b, a) and $(a + b, c + d)$ respectively. Similarly, no lattice points except the vertices lie on the edge joining (b, a) and $(a + c, b + d)$.

Therefore, exactly four points lie on the boundary of this parallelogram.

Now, we apply Pick's Formula, which states that: For any simple polygon with integer vertex co-ordinates having I points in its interior and B points on its boundary, its area A is

$$A = I + \frac{B}{2} - 1.$$

Since the parallelogram in question has four points on its boundary and no interior points, we plug in $I = 0, B = 4$ to find its area.

$$A = 0 + \frac{4}{2} - 1 = 1$$

Hence, the parallelogram has area 1. The area of a parallelogram is $A = bh$ where b is the length of the base and h is height. For this parallelogram, $b = \sqrt{c^2 + d^2}$ and the height is the perpendicular distances between the lines $y = \dfrac{d}{c}x$ and $y = \frac{d}{c}x + b - \frac{da}{c}$ which is $\dfrac{|bc - da|}{\sqrt{c^2 + d^2}}$.

$$A = bh = \sqrt{c^2 + d^2}\frac{|bc - da|}{\sqrt{c^2 + d^2}} = |bc - da| = 1$$

(Q.E.D)

3.2 Proving Hurwitz's Theorem

Now that we have proved the results involving Farey Sequences required for our proof, we can proceed. We prove by contradiction, following the argument presented in [3]

Start by assuming that the statement of Hurwitz's theorem is *false*. Let $a/b, c/d$ be Farey Neighbours and our irrational number, α between them. Let $\dfrac{a+c}{b+d} = \dfrac{e}{f}$. For the sake of contradiction, assume

$$\alpha - \frac{a}{b} \geq \frac{1}{\sqrt{5}b^2}$$

$$\frac{c}{d} \leq \frac{1}{\sqrt{5}d^2}$$

$$\alpha - \frac{e}{f} \geq \frac{1}{\sqrt{5}f^2}$$

Adding the first and third inequality to the second inequality separately, we get

$$\frac{c}{d} - \frac{a}{b} \leq \frac{1}{\sqrt{5}q^2}\left(\frac{1}{b^2} + \frac{1}{d^2}\right)$$

$$\frac{c}{d} - \frac{e}{f} \leq \frac{1}{\sqrt{5}q^2}\left(\frac{1}{d^2} + \frac{1}{f^2}\right)$$

By Property 2 of Farey Neighbours, we know that

$$\frac{c}{d}-\frac{a}{b}=\frac{1}{bd},\frac{c}{d}-\frac{e}{f}=\frac{1}{df}$$

Hence, we can simplify these inequalities into

$$\frac{1}{bd}\geq\frac{1}{\sqrt{5}}\left(\frac{1}{b^2}+\frac{1}{d^2}\right)\Rightarrow\sqrt{5}bd\geq d^2+b^2$$

$$\frac{1}{df}\geq\frac{1}{\sqrt{5}}\left(\frac{1}{d^2}+\frac{1}{f^2}\right)\Rightarrow\sqrt{5}df>d^2+f^2$$

Adding the two inequalities and using the fact that $f=b+d$, we get

$$\sqrt{5}d(b+f)=\sqrt{5}(2b+d)\geq 2d^2+b^2+f^2=3d^2+2b^2+2bd$$

Making the left side to be equal to zero, we get

$$0\geq 3d^2+2b^2+2bd-\sqrt{5}d(2b+d)=\frac{1}{2}((\sqrt{5}-1)d-2b)^2$$

$$\begin{aligned}&\Rightarrow((\sqrt{5}-1)d-2b)^2=0\\&\Rightarrow(\sqrt{5}-1)d-2b=0\\&\Rightarrow\sqrt{5}-1=\frac{2b}{d}\end{aligned}$$

Which implies that $2b/d$ is irrational, a contradiction, as both b,d are integers.

(Q.E.D)

4 Lagrange Numbers and Markov Numbers

Recall how we stated that the $\sqrt{5}$ constant in Hurwitz's theorem can not be improved upon as when $a=\varphi$, any higher multiple of q^2 will result in only *finitely* many $\frac{p}{q}$ satisfying this inequality. This raises a question: if $\alpha\neq\varphi$, what is the biggest multiple of q^2 we can have in the denominator? This question was answered by Hurwitz. Consider

$$\left|\alpha-\frac{p}{q}\right|>\frac{1}{kq^2}$$

Hurwitz showed that if we omit the number φ, we can increase the k to $2\sqrt{2}$. But when we take $2\sqrt{2}$, the number $\alpha=\sqrt{2}$ poses a problem, as for any multiple of q^2 higher than $2\sqrt{2}$ and $\alpha=\sqrt{2}$, we only have finitely many solutions. But now, what if we omit $\alpha=\varphi and\sqrt{2}$? Then we can increase k to $\sqrt{221}/5$. Continuing this process of omission, we get the infinite sequence of numbers $\sqrt{5}$, $2\sqrt{2}$, $\sqrt{221}/5\ldots$ which converge to 3. These numbers are called **Lagrange Numbers.** Let us denote the n-th Lagrange number by $\mathbb{L}_n$

Very closely related to the Lagrange Numbers and Lagrange Spectrum are the **Markov Numbers.**

A Markov Number is a positive integer x, y or z such that it is a part of an positive integral triple that satisfies the below equation, known as the **Markov Diophantine Equation**

$$x^2 + y^2 + z^2 = 3xyz$$

Let us now define the n-th Markov Number m_n to be the n-th smallest number such that the equation

$$m_n^2 + y^2 + z^2 = 3m_nyz$$

has a solution in positive integers for (x, y).

There exists a relation between the n-th Markov Number and Lagrange number. Particularly,

$$\mathbb{L}_n = \sqrt{9 - \frac{4}{m_n^2}}$$

The smallest Markov number is 1, which means that the smallest Lagrange Number is $\sqrt{5}$.

The smaller the Markov number, the smaller the Lagrange number, and hence the *worse* the associated irrational can be approximated. This means that φ is the worst approximable irrational number, as $\sqrt{5}$ is the smallest Lagrange number.

4.1 Lagrange Spectrum

We start with Hurwitz's theorem:

$$|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}$$

We know that $\sqrt{5}$ is the best uniform constant that works for *all* irrationals. But what about individual irrational numbers? Can this constant be improved?

Consider

$$|\alpha - \frac{p}{q}| < \frac{1}{Lq^2}$$

We now define the Lagrange number of a real number α to be the supremum of the set of all L satisfying the above inequality. We denote this by $L(\alpha)$. This leads us to the Lagrange Spectrum — the set of all real numbers L for which there exist *infinitely many* rational numbers $\frac{p}{q}$ for which the above inequality holds. That is,

$$\mathcal{L} = \{L(\alpha) : \alpha \in \mathbb{R}n\mathbb{Q}\}$$

The smallest element of the Lagrange spectrum is $\sqrt{5}$, corresponding to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ — the *most* difficult irrational number to approximate using rationals. Other rationals, like $\sqrt{2}$, can be approximated well, corresponding to larger constants in the Lagrange Spectrum.

The Lagrange spectrum has a very interesting geometry — it is initially a discrete set, in the interval $[\sqrt{5}, 3)$. The last gap in The Lagrange Spectrum is known as Freiman's constant F; namely:

$$F = \frac{2221564096 + 283748\sqrt{462}}{491993569} = 4.52782956\ldots$$

All real numbers in the interval $[F, \infty)$ are part of the Lagrange Spectra. This means that the part of the Lagrange spectrum lying after F is continuous.

Let us look at some pictures of approximations of parts of the Lagrange spectrum

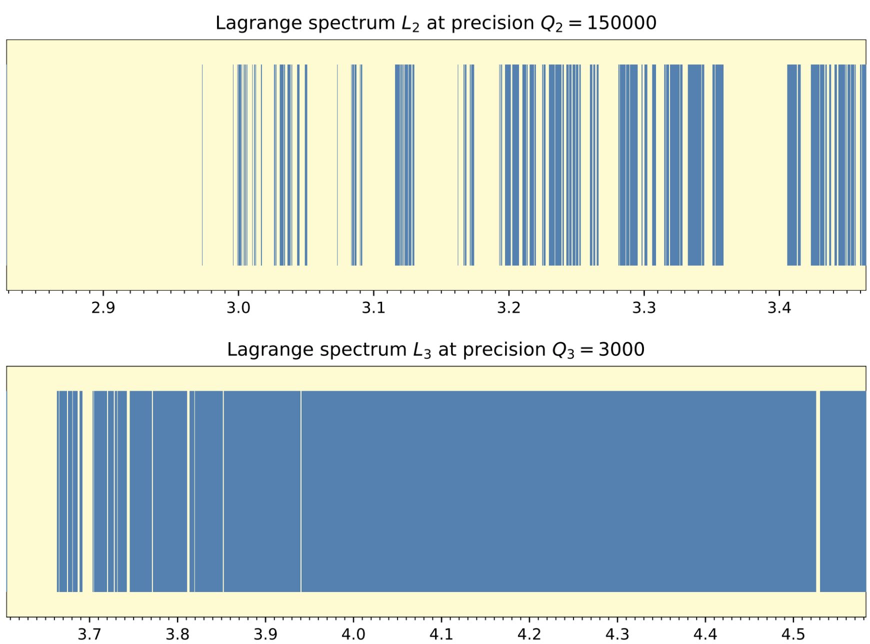


Figure 1: Parts of the Lagrange Spectra visualised, obtained from [2]

5 Continued Fraction

So far, we've talked about how well irrational numbers can be approximated by rational numbers. But we haven't yet addressed a crucial question: how do we actually find good approximations? To answer this, we look to continued fractions.

A continued fraction is an expression of the form:

$$\alpha = b_0 + \cfrac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 \dots}}}$$

Where $b_0, b_1, b_2, b_3, \dots$ and $a_1, a_2, a_3, \dots$ are integers.

We shall be dealing with some special continued fractions, called simple continued fractions. Simple continued fractions are continued fractions where all the numerators are equal to 1. So, an expression of the form

$$\alpha = b_0 + \cfrac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 \dots}}}$$

Generally written as

$$\alpha = [b_0; b_1, b_2, b_3, \dots]$$

Note that if α is an integer, we have $b_0 = \alpha$ and we call it a degenerate case. For non integer α, we can find b_n for $n > 0$ by writing b_{n-1} in the form $\lfloor b_{n-1} \rfloor + \{b_{n-1}\}$, where $\lfloor x \rfloor$ is defined as the greatest integer that *does not* exceed x, rewriting it as $\lfloor b_{n-1} \rfloor + \frac{1}{\frac{1}{\{b_{n-1}\}}}$. $b_n = \lfloor \frac{1}{\{b_{n-1}\}} \rfloor = \lfloor \frac{1}{b_{n-1} - \lfloor b_{n-1} \rfloor} \rfloor$. When $n = 0$, $b_0 = \lfloor \alpha \rfloor$

Note that every b_n for $n > 0$ must be a positive integer as $b_n = \lfloor \frac{1}{\{b_{n-1}\}} \rfloor$ and $0 \leq \{b_{n-1}\} < 1$ which means $\frac{1}{b_{n-1}} > 1$.

Every real number has a simple continued fraction expansion. If the expansion is finite, the real number is also rational. Otherwise, it is irrational.

When we truncate simple continued fractions at some point, we obtain rational approximations called *convergents.* Generally, when truncating the simple continued fractions at $b_0, b_1, b_2, b_3 \dots$ we get the convergents $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \dots$.

A very interesting property of these convergents is that they are recursively defined

Recursive formula for Convergents

Let $\dfrac{p_n}{q_n}$ represent the n-th convergent of some real number $\alpha = [b_0, b_1, b_2 \dots]$

$$p_n = b_n p_{n-1} + p_{n-2}$$

$$q_n = b_n q_{n-1} + q_{n-2}$$

With initial values $p_{-1} = 1,\ p_0 = b_0,\ q_{-1} = 0,\ q_0 = 1.$

Proof of the recursion:

We prove by induction. We can manually verify that the recursion holds true for $n = 1$. Now, we assume it to be true for n.

$$\frac{p_n}{q_n} = \frac{b_n p_{n-1} + p_{n-2}}{b_n q_{n-1} + q_{n-2}}$$

We can get the next convergent p_{n+1}/q_{n+1} by considering the n-th partial quotient to be $b_n + \dfrac{1}{b_{n+1}}$ instead of b_n. This means we can find the $n+1$-th convergent, p_{n+1}/q_{n+1} by replacing b_n with $b_n + \frac{1}{b_{n+1}}$. Performing this replacement, we have

$$\begin{aligned}
\frac{p_{n+1}}{q_{n+1}} &= \frac{(b_n + \frac{1}{b_{n+1}})p_{n-1} + p_{n-2}}{(b_n + \frac{1}{b_{n+1}})q_{n-1} + q_{n-2}} \\
&= \frac{b_n b_{n+1} p_{n-1} + p_{n-1} + b_{n+1} p_{n-2}}{b_n b_{n+1} q_{n-1} + q_{n-1} + b_{n+1} q_{n-2}} \\
&= \frac{b_{n+1}(b_n p_{n-1} + p_{n-2}) + q_{n-1}}{b_{n+1}(b_n q_{n-1} + q_{n-2}) + q_{n-1}} \\
&= \frac{b_{n+1} p_n + p_{n-1}}{b_{n+1} q_n + q_{n-1}}
\end{aligned}$$

(Q.E.D)

Property of Consecutive Convergents

Let $\dfrac{p_n}{q_n}$ represent the n-th convergent of some real number α

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$$

Proof:

$$\begin{aligned}
p_n q_{n-1} - p_{n-1} q_n &= (a_n p_{n-1} + p_{n-2}) q_{n-1} - p_{n-1}(a_n q_{n-1} + q_{n-2}) \\
&= a_n p_{n-1} q_{n-1} + p_{n-2} q_{n-1} - a_n p_{n-1} q_{n-1} - p_{n-1} q_{n-2} \\
&= p_{n-2} q_{n-1} - p_{n-1} q_{n-2} \\
&= -(p_{n-1} q_{n-2} - p_{n-2} q_{n-1})
\end{aligned}$$

Repeating this step with $n-1, n-2, \ldots, 2$ in place of n, we get:

$q_0 = 1, q_1 = b1 > 0, q_2 = b_2 q_1 + q_0 > 0, \ldots$

5.1 Error of approximation for the n-th convergent

A key property of the convergents is the following bound for the error of the Convergents

Error Bound of m-th convergent

Let $\dfrac{p_n}{q_n}$ represent the n-th convergent of some real number α.

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

Proof: Notice that the n-th convergent of $\alpha = [b_0, b_1, b_2 \ldots]$ can be written as

$$\frac{p_n}{q_n} = \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} + \frac{p_{n-1}}{q_{n-1}} - \frac{p_{n-2}}{q_{n-2}} + \frac{p_{n-2}}{q_{n-2}} - \frac{p_{n-3}}{q_{n-3}} + \frac{p_{n-3}}{q_{n-3}} \cdots + \frac{p_{m+1}}{q_{m+1}} - \frac{p_m}{q_m} + \frac{p_m}{q_m}$$

The above tells us that α can be written in the form

$$\alpha = \frac{p_0}{q_0} - \frac{p_0}{q_0} + \frac{p_1}{q_1} - \frac{p_1}{q_1} + \frac{p_2}{q_2} - \frac{p_2}{q_2} + \frac{p_3}{q_3} - \frac{p_3}{q_3} \ldots$$

Which in turn tells us that α can be written in the form

$$\alpha = k_0 - k_1 - k_2 - k_3 \ldots$$

Where $k_n = k_n = \dfrac{p_{n-1}}{q_{n-1}} - \dfrac{p_n}{q_n}, n > 1$. These k_n become smaller and smaller, and α minus the first n terms is less than the value of k_{n+1}. This directly implies the inequality.

(Q.E.D)

This shows that convergents provide rational approximations that are far better than those guaranteed by Dirichlet's theorem alone. In fact, the n-th convergent of α, $\frac{p_n}{q_n}$ is the best rational approximation of α with denominator at most q_n. A simple proof of this is below.

Proof: Consider a fraction a/b that is closer to α than its convergent p_n/q_n.That is,

$$\left|\alpha - \frac{a}{b}\right| < \left|\alpha - \frac{p_n}{q_n}\right|$$

Case 1: Both $\dfrac{a}{b}, \dfrac{p_n}{q_n}$ lie on same side of α.

In this case, a/b must lie between α and p_n/q_n. That is,

$$\frac{1}{q_n^2} \geq \frac{1}{q_n q_{n+1}} > \left|\alpha - \frac{p_n}{q_n}\right| \geq \left|\frac{a}{b} - \frac{p_n}{q_n}\right| = \frac{|aq_n - bp_n|}{bq_n} \geq \frac{1}{bq_n} \Rightarrow b > q_n$$

Case 2: Both $\dfrac{a}{b}, \dfrac{p_n}{q_n}$ lie on different sides of α.

In this case, p_n/q_n lies between a/b and α

$$\frac{1}{q_n^2} \geq \frac{1}{q_n q_{n+1}} > \left|\alpha - \frac{p_n}{q_n}\right| \geq \left|\frac{a}{b} - \frac{p_n}{q_n}\right| = \frac{|ap_n - bq_n|}{bq_n} \geq \frac{1}{bq_n} \Rightarrow b > q_n$$

(Q.E.D)

Let us now look at the error of approximation of the convergents of some irrational numbers.

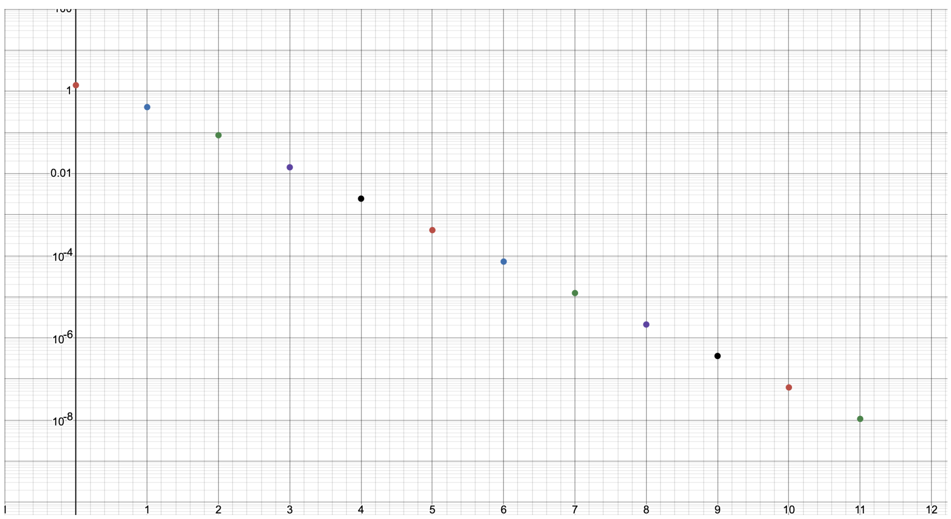


Figure 2: Error of approximation for the convergents of $\sqrt{2}$

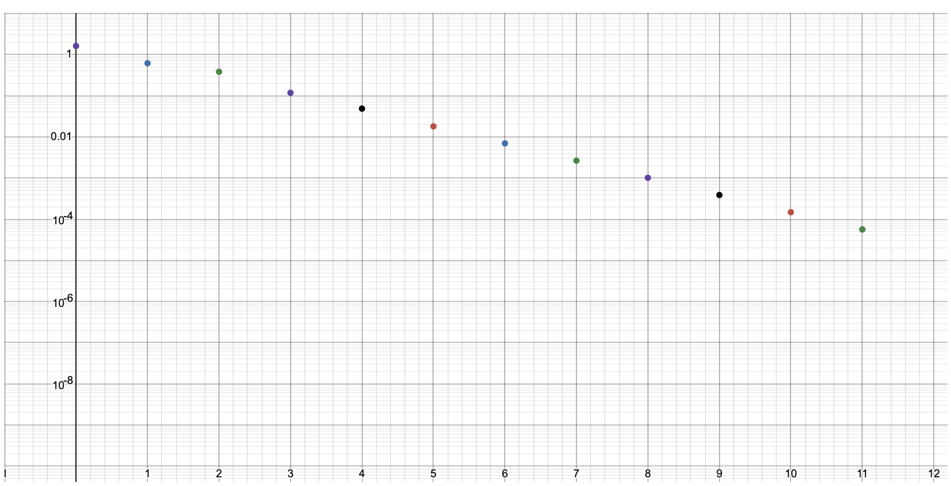


Figure 3: Error of approximation for the convergents of φ

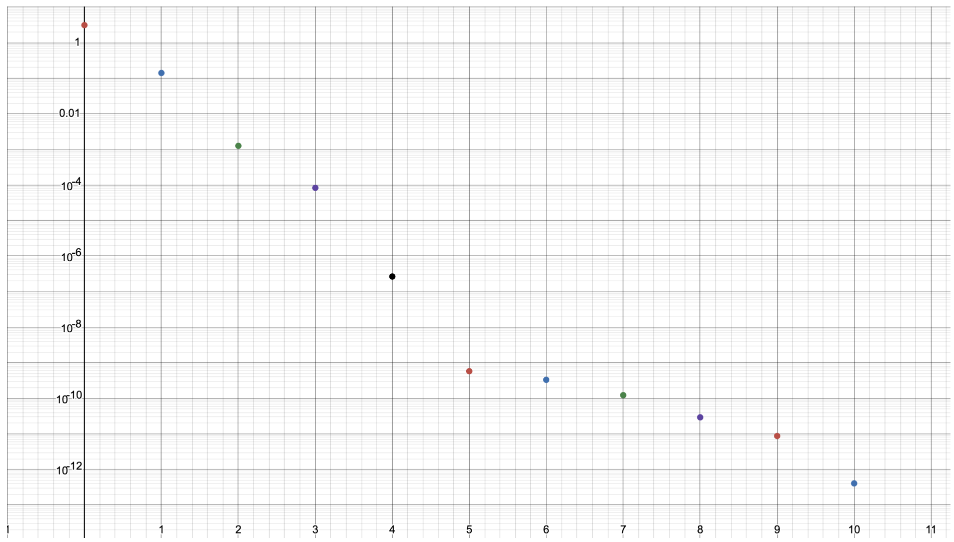


Figure 4: Error of approximation for the convergents of π

Note that for these graphs are scaled logarithmically in the y- axis with points at $(n, |\alpha - p_n/q_n|)$ Observe how the error of approximation of π is nearly 4 orders of magnitude less than that of $\sqrt{2}$ for the same convergents. This indicates that π is more well approximable than $\sqrt{2}$ and φ.

Additionally, notice how the error of approximation decreases much faster for $\sqrt{2}$ than for φ — displaying how $\sqrt{2}$ is far more accurately approximable using rational numbers than φ.

5.2 Quadratic Irrationals

A Quadratic Irrational is a real number that is a root of a quadratic equation with integer coefficients, but is not rational. They are real numbers of the form $a + b\sqrt{n}$ where $a, b \in Q, b \neq 0$ where n is not the square of any integer. Interestingly enough, these are the only types of irrationals that have eventually periodic continued fraction representations. Note that all quadratic irrationals will be the roots of a quadratic equation with integer coefficients.

Proving that if the continued fraction expansion of α is eventually periodic, it is a quadratic irrational.

Let $\alpha = [b_0, b_1, \ldots \overline{b_{k+1}, b_{k+2}, \ldots, b_{k+n}}]$ where $[\overline{b_{k+1}, b_{k+2}, \ldots, b_{k+n}}]$ represents a continued fraction where $b_{x+mn} = b_x$ for all $m \in \mathbb{N}$. Let $\beta = [\overline{b_{k+1}, b_{k+2}, \ldots, b_{k+n}}]$

Since the expression is periodic, we have $\beta = [b_{k+1}, b_{k+2}, \ldots, b_{k+n}, \beta]$

Using the recursion, we have $\beta = \frac{p_{n-1}\beta + p_{n-2}}{q_{n-1}\beta + q_{n-2}}$. This gives us a quadratic in β. Since β is irrational, this means that β is a quadratic irrational.

$\alpha = [b_0, b_1, \dots \beta]$ is a linear functional transformation of β. This means that $\alpha = \frac{c\beta+d}{e\beta+f}$ where $c, d, e, f \in \mathbb{Z}$. Since β is a quadratic irrational, this means that α is also a quadratic irrational.

Let us now prove the converse. We follow the argument presented in [4] Assume β is a quadratic irrational and

$$\beta = [b_0, b_1, b_2, b_3 \dots]$$

Note that $[0, 0, \dots, 0, z]$ in general is greater than 1 if the number of zeros is even and lesser than 1 if the number of zeros is odd. If $[0, 0, \dots b_i, b_{i+1} \dots] = [0, 0, \dots, b_j, b_{j+1}, \dots]$, $i - j$ is even and $b_{i+k} = b_{j+k}$ for all k.

There exists $a\beta^2 + b\beta + c = 0$. We denote this assertion by $\beta \in (a, b, c)$. Now, we define the function

$$f(\beta) = \begin{cases} \beta - 1, & \text{if } \beta \geq 1 \\ \dfrac{\beta}{1-\beta}, & \text{if } \beta < 1 \end{cases}$$

Define $\beta_1 = \beta$, and $\beta_{n+1} = f(\beta_n)$ for $n > 1$. We manually verify that

$$a(\beta - 1)^2 + (2a + b)(\beta - 1) + (a + b + c) = a\beta^2 + b\beta + c = 0$$

And

$$(a + b + c)\beta^2 + (b + 2c)\beta(1 - \beta) + c(1 - \beta)^2 = a\beta^2 + b\beta + c = 0,$$

Therefore, $f(\beta) \in (a, 2a + b, a + b + c)$ or $f(\beta) \in (a + b + c, b + 2c, c)$. $(\beta_1, \beta_2 \dots)$ is a sequence of quadratic irrationals and hence determines an infinite sequence of triples $\beta_n \in (a_n, b_n, c_n)$. We assume $b_n > 0$ without loss of generality as $\beta_n \in (a_n, b_n, c_n) \Leftrightarrow \beta_n \in (-a_n, -b_n, -c_n)$.

$$(2a + b)^2 - 4a(a + b + c) = b^2 - 4ac = (b + 2c)^2 - 4c(a + b + c)$$

That is, $b_n^2 - 4a_nc_n$ is independent of n.

If only finitely many triplets (a_n, b_n, c_n) have $c_n < 0$, eventually $a_n, c_n > 0$. Because $\beta_n > 0$, we will have $b_n < 0$. This, however, is impossible as $a_n - b_n + c_n$ would then be strictly decreasing and non-negative. Therefore, $a_nc_n < 0$ *infinitely* often. Now, since $b_n^2 - 4a_nc_n$ is constant, there *must* be a triple which appears thrice in the sequence (a_n, b_n, c_n). Therefore, $\beta_i = \beta_j, i > j$ for some i, j. If $\beta = [b_0, b_1, b_2, b_3 \dots]$, β_i is of the form $[0, \dots, 0, b, b_n, b_{n+1} \dots]$ and $\beta_j = [0, \dots 0, c, b_m, b_{m+1}, \dots]$ where $b, c > 0$ and $n > m$ necessarily. Since $\beta_i = \beta_j$, we have that $n - m$ is a positive even integer, $b = c$ and $b_{m+k} = b_{n+k}$ for all k. Therefore, the sequence b_k is eventually periodic. Which, in turn, means that the simple continued fraction expansion β is also eventually periodic.

(Q.E.D)

Let us now look at an example of a quadratic irrational

Example: $\sqrt{2}$

$$\begin{aligned} &1 < \sqrt{2} < 2 \\ \Rightarrow &\lfloor \sqrt{2} \rfloor = 1 \\ \Rightarrow &\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \frac{1}{1 + \sqrt{2}} \\ \Rightarrow &\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2} \cdots}} \\ \Rightarrow &\sqrt{2} = [1, \overline{2}] \end{aligned}$$

The periodicity of the continued fraction expansion of a quadratic irrational has important consequences. Since the expansion eventually repeats, the number can be described completely by a finite amount of data. This makes it possible to compute arbitrarily many convergents algorithmically. In other words, the number is not just well-approximable; they are computable with precise instructions for generating its rational approximations.

Moreover, this periodicity characterises quadratic irrationals among all real numbers: a real number has an eventually periodic continued fraction expansion if and only if it is the solution to a quadratic equation with integer coefficients. This tells us that quadratic irrationals can be approximated very well using rationals, but in a highly structured, algebraic way.

This begs the question: Are there irrationals that can be approximated better? Or worse? How can we measure how well a number can be approximated by rationals?

6 Measuring Irrationality

Not all irrational numbers are created equal. As we observed earlier, some irrationals, like π, can be approximated well using rationals. Others, like the golden ratio $\varphi = \frac{1 + \sqrt{5}}{2}$ are *extremely* resistant to approximation. This motivates us to ask: Why are certain irrational numbers easier to approximate using rationals? Is there a way to quantify how well or how poorly an irrational number can be approximated using rationals?

6.1 Transcendence and Approximation

A real number is called *algebraic* if it is a root of a non-zero polynomial with integer coefficients. Otherwise, it is *transcendental*. All rational numbers are algebraic, and so are familiar irrationals like $\sqrt{2}$. But numbers like e and π are not roots of any such polynomial; they are transcendental.

If α is an algebraic irrational number, we define its degree d to be the degree of its minimal polynomial. That is, the smallest possible degree of a polynomial f such that $f(\alpha) = 0$.

Earlier, we encountered the Lagrange spectrum, which deals with the best constant c such that $\left|x - \frac{p}{q}\right| < \frac{c}{q^2}$ holds for infinitely many rational numbers. That framework focuses on specific constants for error decay. In contrast, we now shift to a scale-based view of approximation, where we ask how fast the error can decrease, as a function of q. This leads us to the concept of the *irrationality exponent.*

6.2 Liouville-Roth Irrationality Measure

The **irrationality exponent** (also called the **Liouville–Roth measure**) of a real number α, denoted $\mu(\alpha)$, is defined as the supremum of all real numbers μ for which the inequality

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\mu}$$

has infinitely many solutions in coprime integers p and $q > 0$.

For any rational number α, we have $\mu(\alpha)$=1, as a consequence of Dirichlet's approximation theorem.

Note that a higher irrationality exponent means that the irrational number is better approximable.

For any irrational number α, $\mu(\alpha) \geq 2$.

If α is an *algebraic* irrational, that is, it is a real root of a polynomial with integer coefficients, $\mu(\alpha) = 2$. If α is not algebraic, that is, it is transcendental, we have $\mu(\alpha) \geq 2$.

The golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ is known to be the worst approximable irrational number. Among all irrationals, it has the smallest possible irrationality exponent—exactly 2. Additionally, its continued fraction expansion is $[1; 1, 1, 1, \dots]$, which leads to the *slowest* convergence of its convergents, making it maximally resistant to rational approximation. We shall reinforce the notion of φ's poor approximability using another measure of irrationality later on.

For most transcendental numbers, their irrationality measure is exactly 2. For others, the exact value of their irrationality exponent is not known. Below are some transcendental numbers along with the known upper and lower bounds of their irrationality exponent.

Number α	Exact $\mu(\alpha)$	Lower Bound	Upper Bound
e	2	2	2
$\ln 2$	unknown	2	3.57455...
$\ln 3$	unknown	2	5.11620...
π	unknown	2	7.10320...
π^2	unknown	2	5.09541...

There exists a special class of numbers, called **Liouville Numbers**, that have irrationality measure ∞. That is, they can be approximated to rationals exceptionally well.

6.3 Liouville Numbers

We now start by stating Liouville's theorem - a tool that helps us differentiate between the approximability of algebraic irrational numbers and transcendental numbers.

Liouville's Theorem

Let $\alpha \in \mathbb{R}$ be an irrational number with degree d. Then, there is a non-zero constant C such that for every rational number $\frac{p}{q} \in \mathbb{Q}$,

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C}{q^d}$$

We follow the argument presented in [6]

Proof:

Let $r_1, r_2, \ldots, r_k$ be the rational roots of a polynomial f of degree n that has α as a root. Since α is irrational, $r_i \neq \alpha \, \forall i$

Let $c_1 > 0$ be the minimum of $|\alpha - r_i|$. If there are no r_i, let $c_1 = 1$.

Now let $\beta = \dfrac{p}{q}$, where $\beta \notin \{r_1, \ldots, r_k\}$. Then:

$$\begin{aligned} f(\beta) &\neq 0 \\ \Rightarrow |f(\beta)| &\geq \frac{1}{q^n} && \text{Since } f(\beta) \text{ is a rational number with denominator } q^n \\ \Rightarrow |f(\alpha) - f(\beta)| &\geq \frac{1}{q^n} && \text{because } f(\alpha) = 0 \end{aligned}$$

Let

$$f(\alpha) = \sum_{k=0}^{n} a_k \alpha^k$$

$$
\begin{aligned}
\Rightarrow f(\alpha) - f(\beta) &= \sum_{k=0}^{n} a_k \alpha^k - \sum_{k=0}^{n} a_k \beta^k \\
&= \sum_{k=0}^{n} a_k(\alpha^k - \beta^k) \\
&= \sum_{k=1}^{n} a_k(\alpha^k - \beta^k) \qquad \text{since } \alpha^0 - \beta^0 = 0 \\
&= \sum_{k=1}^{n} a_k(\alpha - \beta) \sum_{i=0}^{k-1} \alpha^{k-1-i}\beta^i \\
&= (\alpha - \beta) \sum_{k=1}^{n} a_k \sum_{i=0}^{k-1} \alpha^{k-1-i}\beta^i
\end{aligned}
$$

Case 1: $|\alpha - \beta| \leq 1$

Then:

$$
\begin{aligned}
|\beta| - |\alpha| &\leq |\alpha - \beta| \\
\Rightarrow |\beta| &\leq |\alpha| + 1
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
|f(\alpha) - f(\beta)| &\leq |\alpha - \beta| \sum_{k=1}^{n} |a_k| \sum_{i=0}^{k-1} |\alpha^{k-1-i}\beta^i| \qquad \text{Triangle Inequality} \\
&\leq |\alpha - \beta| \sum_{k=1}^{n} |a_k| \sum_{i=0}^{k-1} |\alpha^{k-1-i}(1 + |\alpha|)^i| \qquad \text{substituting } \alpha = |x| + 1 \\
&\leq |\alpha - \beta| \sum_{k=1}^{n} |a_k| \sum_{i=0}^{k-1} |\alpha^{k-1} \left(\tfrac{|\alpha|+1}{|\alpha|}\right)^i | \\
&\leq |\alpha - \beta| \sum_{k=1}^{n} |a_k \alpha^{k-1}| \sum_{i=0}^{k-1} \left(1 + \frac{1}{|\alpha|}\right)^i \qquad \text{moving } \alpha^{k-1} \text{ out of the nested sum} \\
&\leq |\alpha - \beta| \sum_{k=1}^{n} |a_k \alpha^{k-1}| \cdot \frac{\left(1 + \frac{1}{|\alpha|}\right)^k - 1}{\left(1 + \frac{1}{|\alpha|}\right) - 1} \\
&= |\alpha - \beta| \sum_{k=1}^{n} |a_k \alpha^k| \left(\left(1 + \frac{1}{|\alpha|}\right)^k - 1\right) \\
&\leq |\alpha - \beta| \sum_{k=1}^{n} |a_k| \left((|\alpha| + 1)^k - |\alpha|^k\right)
\end{aligned}
$$

Let

$$
c_\alpha = \sum_{k=1}^{n} |a_k| \left((|\alpha| + 1)^k - |\alpha|^k\right)
$$

$$\Rightarrow |f(\alpha) - f(\beta)| \leq |\alpha - \beta| \cdot c_\alpha$$
$$\Rightarrow |\alpha - \beta| \geq \frac{|f(\alpha) - f(\beta)|}{c_\alpha} \geq \frac{1}{c_\alpha q^n}$$

Case 2: $|\alpha - \beta| > 1$

Then:

$$|\alpha - \beta| > 1 \geq \frac{1}{q^n}$$

(Q.E.D)

As a Corollary of the above, we have the following result.

Corollary of Liouville's Theorem

For every transcendental number α, $C > 0$, $d \geq 1$, there exists a rational number $\frac{p}{q} \in \mathbb{Q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{C}{q^d}$$

This tells us that transcendental numbers are far more well-approximable than algebraic irrational numbers. Yet, not all transcendental numbers are created equal, either. Among the transcendentals, **Liouville numbers** are so well-approximated that their irrationality exponent is infinite.

A **Liouville number** is a real number α such that, for every positive integer n, there exist integers p and $q > 1$ satisfying

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}$$

This means the error in approximating α can decay faster than *any* power of $1/q$ - an extreme property not shown by any algebraic numbers.

Joseph Liouville used this property to construct the first known decimal example of a transcendental number. An explicit example of a Liouville number is

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{10^{k!}} = 0.110001000000000000000001\ldots$$

The gaps of zeros between the 1s grow rapidly, which ensures the number can be approximated extremely well by truncating the sum at various stages.

All Liouville numbers are transcendental, but not all transcendental numbers are Liouville numbers. For instance, e and π are transcendental, but their irrationality exponents have known exact values or upper bounds. Thus, Liouville numbers form a very special subclass of transcendental numbers, distinguished by their exceptional approximability.

6.4 Markov Constant and the Most Irrational Number

Another way to measure how irrational a number is comes from looking at how *badly* it can be approximated by rationals. Instead of asking how small the error $|\alpha - \frac{p}{q}|$ can get, we flip the question: what's the best lower bound we can *guarantee* for that error?

The Markov constant of a real number α, denoted $M(\alpha)$, is defined as:

$$M(\alpha) = \sup \left\{ M > 0 \;\middle|\; \left|\alpha - \frac{p}{q}\right| < \frac{1}{Mq^2} \text{ for infinitely many } \frac{p}{q} \in \mathbb{Q} \right\}$$

Note that $M(\alpha)$ being higher implies α is well approximable using the rationals. If the set does not have an upper bound, $M(\alpha) = \infty$.

So, if $M(\alpha)$ is small, then no matter how hard you try, rational approximations $\frac{p}{q}$ will always miss α by at least $\frac{1}{Mq^2}$, for all large enough q. That makes α hard to approximate - in a sense, very irrational.

$M(\alpha)$ is not continuous, as it is undefined for rationals. The maxima and minima of $M(\alpha)$ are ∞ and $\sqrt{5}$ respectively. Here's the neat part:

$$M(\varphi) = \sqrt{5}$$

This tells us that φ is the *worst* approximable irrational number. This observation can also be justified by looking at the convergence of φ.

$$\varphi = [1; 1, 1, 1, 1, \dots]$$

All the partial quotients are as small as possible - all 1s - which makes the denominators grow slowly, that is, error decays slowly and prevents any rational from sneaking up on φ too close too quick.

Even better: any number that's a linear functional transformation in φ - that is, any $\frac{a\varphi+b}{c\varphi+d}$ with integers a, b, c, d and $ad - bc \neq 0$ - will have the same Markov constant. This is because every number of such form can be written in the form $\alpha\varphi + \beta$, where

$\alpha, \beta \in \mathbb{Q}$. A brief proof is as below.

$$
\begin{aligned}
\frac{a\varphi + b}{c\varphi + d} &= \frac{a\frac{1+\sqrt{5}}{2} + b}{c\frac{1+\sqrt{5}}{2} + d} \\
&= \frac{(1+\sqrt{5})a + 2b}{(1+\sqrt{5})c + 2d} \\
&= \frac{(1+\sqrt{5})c + 2b}{\sqrt{5}c + (c+2d)} \\
&= \frac{[(1+\sqrt{5})c + 2b][\sqrt{5}c - (c+2d)]}{5c^2 - c^2 - 4d^2 - 4dc} \\
&= \frac{[(1+\sqrt{5})c + 2b][(1+\sqrt{5})c - (2c+2d)]}{4c^2 - 4d^2 - 4dc} \\
&= \frac{4c^2\varphi^2 + 4c(b-c-d)\varphi - 4b(c+d)}{4c^2 - 4d^2 - 4dc} \\
&= \frac{4c^2(\varphi+1) + 4c(b-c-d)\varphi - 4b(c+d)}{4c^2 - 4d^2 - 4dc} \\
&= \frac{4c(b-d)\varphi + 4c^2 - 4b(c+d)}{4c^2 - 4d^2 - 4dc} \\
&= \frac{c(b-d)\varphi + c^2 - b(c+d)}{c^2 - d^2 - dc} \\
&= \frac{c(b-d)}{c^2 - d^2 - dc}\varphi + \frac{c^2 - b(c+d)}{c^2 - d^2 - dc}
\end{aligned}
$$

Which means that any rational expression in φ can be written as a linear combination of φ in terms of rational coefficients. That is, any rational expression in φ is as well approximable as φ.

So if Liouville numbers live on one end of the spectrum - being way too friendly with rationals - then the golden ratio lives proudly on the other end, as the most unapproachable irrational number out there.

Now, recall an earlier observation we made while observing graphs displaying the error of approximation for the convergence of different irrational numbers: The error of approximation decreases extremely slowly for φ, and extremely fast for π in comparison to that of $\sqrt{2}$. Turns out that we need *both* Liouville and Markov's findings to appropriately justify these observations. The corollary of Liouville's theorem tells us that transcendental numbers like π are far more well-approximable than algebraic irrational numbers like $\sqrt{2}$, and the Markov constant of φ tells us that it is the worst approximable irrational number.

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