Orthogonal Polynomials

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Euler Circle

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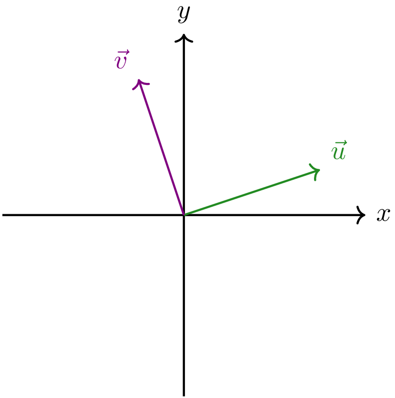
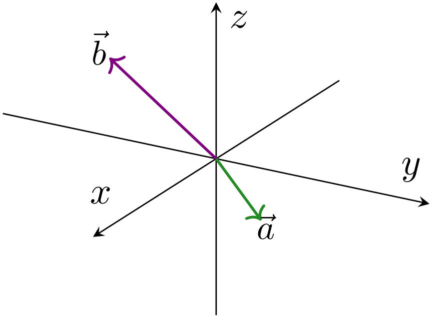
Vectors



2D Orthogonal Vectors



3D Orthogonal Vectors

Dot Product

Definition (Geometric)

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

- $\vec{u}, \vec{v}$ are vectors
- $\|\vec{u}\|$, $\|\vec{v}\|$ are magnitudes
- θ is the angle between them

Definition (Component Form)

If $\vec{u}, \vec{v} \in \mathbb{R}^n$ with components
$\vec{u} = (u_1, u_2, \ldots, u_n), \vec{v} = (v_1, v_2, \ldots, v_n)$, then:

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i$$

Vector Spaces

Definition

A **vector space** V over a field $\mathbb{F}$ (like $\mathbb{R}$) is a set with:

- vector addition: $V \times V \to V$
- scalar multiplication: $\mathbb{F} \times V \to V$

that satisfies the following axioms for all $\vec{u}, \vec{v}, \vec{w} \in V$, and $a, b \in \mathbb{F}$:

Addition

A1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

A2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

A3. $\exists \vec{0} \in V : \vec{v} + \vec{0} = \vec{v}$

A4. $\exists -\vec{v} \in V : \vec{v} + (-\vec{v}) = \vec{0}$

Scalar Multiplication

A5. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

A6. $(a + b)\vec{v} = a\vec{v} + b\vec{v}$

A7. $a(b\vec{v}) = (ab)\vec{v}$

A8. $1\vec{v} = \vec{v}$

Examples of Vector Spaces

- $\mathbb{R}^n$: Euclidean space of n-tuples of real numbers
- $\mathbb{C}$: The field of complex numbers (also a vector space over $\mathbb{R}$)
- $\mathcal{F}(X, \mathbb{R})$: Space of all real-valued functions defined on a set X
- $\mathbb{R}[x]$: Space of all real polynomials in variable x
- $M_{m \times n}(\mathbb{R})$: Space of all $m \times n$ real matrices

Inner Products

Definition

An **inner product** on a vector space V over $\mathbb{R}$ is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

satisfying, for all $\vec{u}, \vec{v}, \vec{w} \in V$, and $a \in \mathbb{R}$:

- $\langle \vec{v}, \vec{v} \rangle \geq 0$, with equal to 0 if.f. $\vec{v} = \vec{0}$
- $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
- $\langle a\vec{u} + \vec{w}, \vec{v} \rangle = a\langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$

Example: Function Space Inner Product

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)\,dx \quad \text{with } w(x) > 0$$

Orthogonal Polynomials

Definition

A sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ is called **orthogonal** with respect to a weight function $w(x) > 0$ on an interval $[a, b]$ if for all $m \neq n$:

$$\int_a^b P_m(x)P_n(x)w(x)\,dx = 0$$

and each $P_n(x)$ is of degree n.

Legendre Polynomials

Definition

The Legendre polynomials $\{P_n\}$ are orthogonal on $[-1, 1]$ with weight $w(x) = 1$:

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0, \quad m \neq n.$$

First few polynomials:

$$P_0 = 1, \quad P_1 = x, \quad P_2 = \frac{1}{2}(3x^2 - 1), \quad P_3 = \frac{1}{2}(5x^3 - 3x).$$

Laplace's Equation

$$\nabla^2 \Phi = 0$$

Legendre polynomials arise when solving this in spherical coordinates.

Legendre's DE

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0,$$

with solutions $y = P_n(x)$.

Chebyshev Polynomials

Definition (Orthogonality)

The Chebyshev polynomials of the first kind $\{T_n(x)\}$ are orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = \dfrac{1}{\sqrt{1-x^2}}$:

$$\int_{-1}^{1} T_m(x) T_n(x) \frac{1}{\sqrt{1-x^2}}\, dx = 0 \quad \text{for } m \neq n.$$

First few Chebyshev polynomials:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x$$

Trigonometric Definition

$$T_n(x) = \cos(n \arccos x)$$

Hermite Polynomials

Orthogonality

Hermite polynomials $H_n(x)$ are orthogonal on $(-\infty, \infty)$ with weight $w(x) = e^{-x^2}$:

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx = 0 \quad \text{for } m \neq n.$$

First few Hermite polynomials:

$$H_0 = 1, \quad H_1 = 2x, \quad H_2 = 4x^2 - 2, \quad H_3 = 8x^3 - 12x$$

Hermite Differential Equation

$$y'' - 2xy' + 2ny = 0$$

Quantum Harmonic Oscillator

The Schrödinger equation:

$$-\frac{\hbar^2}{2m}\psi'' + \frac{1}{2}m\omega^2 x^2\psi = E\psi$$

has solutions:

$$\psi_n(x) = C_n H_n(x)e^{-x^2/2}$$

Gram-Schmidt on Polynomial Space

Given a basis $\{1, x, x^2, \dots\}$, we can orthonormalize it using:

$$P_n(x) = x^n - \sum_{k=0}^{n-1} \frac{\langle x^n, P_k \rangle}{\langle P_k, P_k \rangle} P_k(x)$$

with inner product $\langle f, g \rangle = \int_a^b f(x) g(x) w(x)\, dx$

Recurrence Relations

- **Legendre:** $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$
- **Chebyshev:** $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$
- **Hermite:** $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$
- **Laguerre:** $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$

Rodrigues' / Derivative Formulas

- **Legendre:** $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$
- **Hermite:** $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$

Approximation

Fourier Series

A 2π-periodic function $f(x)$ can be written as:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

with:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

Orthogonal Polynomial Approximation

Given orthogonal polynomials $P_n(x)$ on $[a, b]$ with weight $w(x)$:

$$f(x) \approx \sum_{n=0}^{N} a_n P_n(x), \quad a_n = \frac{\int_a^b f(x) P_n(x) w(x)\, dx}{\int_a^b P_n^2(x) w(x)\, dx}$$

Mean Square Error

$$\text{MSE} = \int_a^b |f(x) - \tilde{f}(x)|^2 w(x)\, dx$$

Roots

Interlacing Roots

A sequence of polynomials $\{P_n(x)\}$ is said to have **interlacing roots** if the real roots of $P_{n-1}(x)$ lie strictly between the real roots of $P_n(x)$.

In other words, if the roots of $P_n(x)$ are:

$$x_1 < x_2 < \cdots < x_n,$$

then the roots of $P_{n-1}(x)$ are:

$$y_1 < y_2 < \cdots < y_{n-1},$$

with

$$x_1 < y_1 < x_2 < y_2 < \cdots < x_{n-1} < y_{n-1} < x_n.$$

Thank You!