Counting Regions in Hyperplane Arrangements

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Euler Circle

Overview

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Hyperplane Arrangement

Definition

A hyperplane be an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$ of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = k.$$

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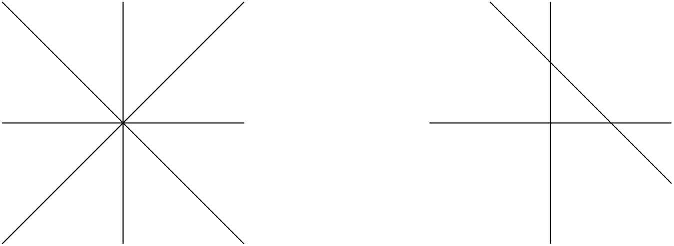

Examples hyperplane arrangements in $\mathbb{R}^2$.

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For a hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_k\}$. A region is a connected component of

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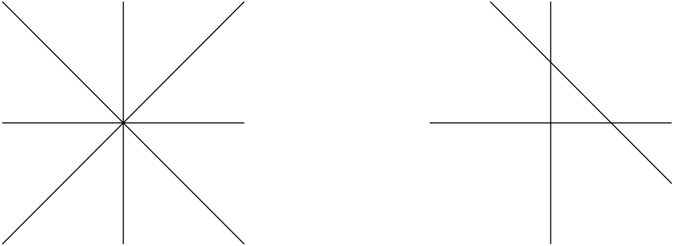

Left arrangement has 8 regions. Right arrangement has 7 regions.

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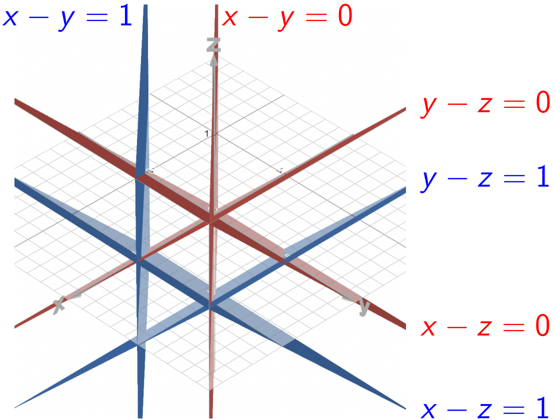

Visualization of $\mathcal{S}_3$.

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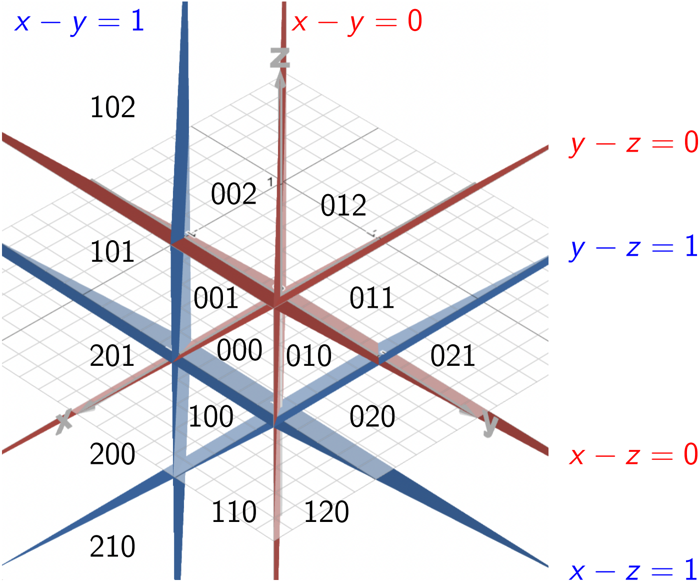
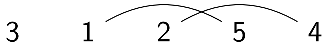
Proposition

There are $(n+1)^{n-1}$ parking functions of length n.

Pak-Stanley Labeling λ



Proof Sketch



corresponds to

$$x_3 > x_1 > x_2 > x_5 > x_4,$$

$$x_1 < x_5 + 1, x_2 < x_4 + 1, x_3 > x_1 + 1, x_1 > x_4 + 1.$$

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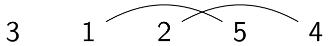

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λ has a corresponding definition in permutations and arcs. The definition implies λ maps to parking functions.

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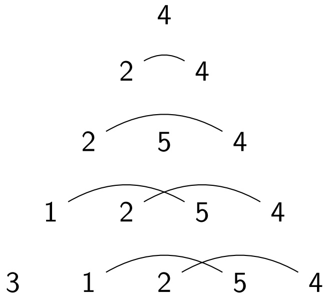
Construct λ^{-1} recursively, e.g, for $(1, 0, 4, 0, 1)$,



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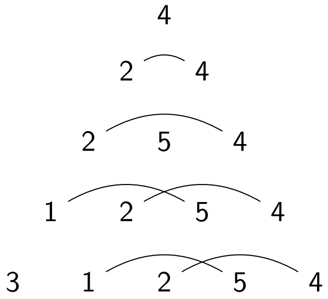

Finally, we show λ^{-1} exists for all parking functions and is unique (very tedious).

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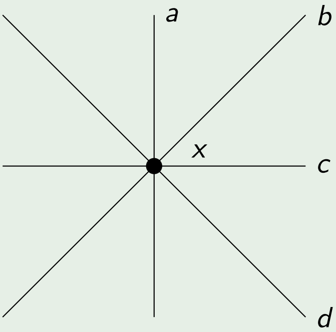
Remove any empty intersections. Replace all intersections with t^d where d is their dimension. The result polynomial $\chi_{\mathcal{A}}(t)$ is the characteristic polynomial.

Characteristic Polynomial

Example



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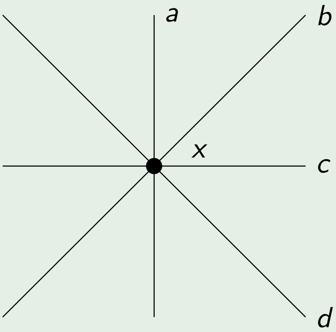

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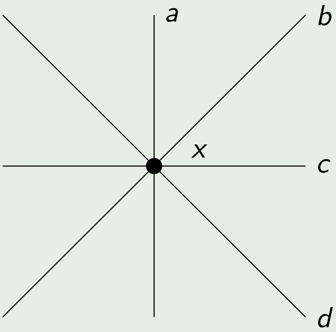

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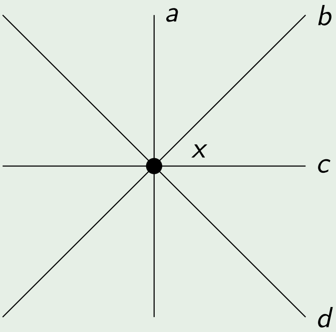

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Theorem (Zaslavsky)

The number of regions formed by an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is $(-1)^n \chi_{\mathcal{A}}(-1)$.

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- For large enough q, an intersection of dimension d contains q^d points.
- $\left| \mathbb{F}_q^n - \bigcup_{H^q \in \mathcal{A}^q} H^q \right| = q^n - \sum q^{\dim(H_i^q)} + \sum q^{\dim(H_i^q \cap H_j^q)} - \cdots = \chi_{\mathcal{A}}(q)$.

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Counting Problem

$$\chi_{\mathcal{S}_n}(q) = |\{(x_1, \ldots, x_n) \in \mathbb{F}_q^n : x_i - x_j \not\equiv 0, 1 \text{ for } 1 \leq i < j \leq n\}|.$$

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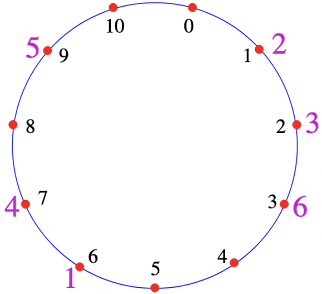

Example for $(6, 1, 2, 7, 9, 3)$.

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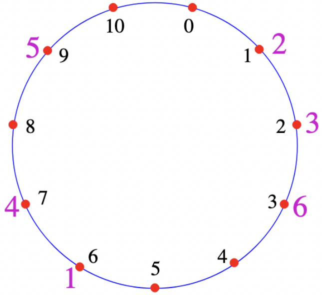

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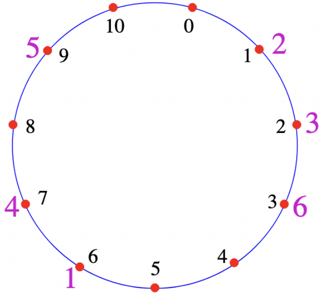

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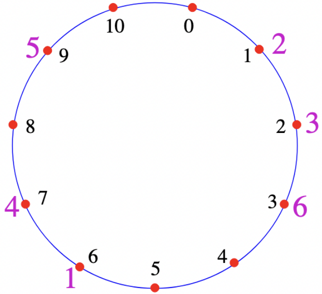

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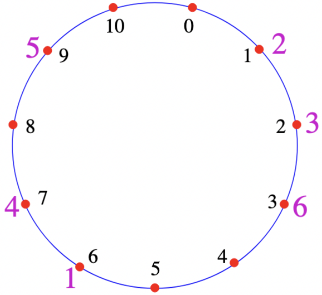

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- Place the balls in B_1, consecutively after 1 in ascending order. Skip a spot. Place balls in B_2 in ascending order. Skip a spot. Continue.
- This is a bijection to placing the balls. Thus, $\chi_{\mathcal{S}_n}(q) = q(q-n)^{n-1}$.

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Number of Regions

So by Zaslavsky's Theorem,

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This method is generalizable.

Question

Find the number of regions in the Catalan arrangement.

$$\mathcal{C}_n = \{x_i - x_j = 0, \pm 1 : 1 \leq i < j \leq n\}.$$

Acknowledgements

Thank you to Simon for suggesting hyperplane arrangements and organizing the IRPW program. Thank you to Emma Cardwell for her advise and support. Thank you to all of you for listening.

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