COMBINATORICS BEHIND HYPERPLANE ARRANGEMENTS

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Abstract. A hyperplane arrangement $\mathcal{A}$ is a set of hyperplanes in $\mathbb{R}^n$. We examine the number of regions $r(\mathcal{A})$ in these arrangements by giving a formula for $r(\mathcal{A})$ in terms of the characteristic polynomial of $\mathcal{A}$, determining the number of regions in the Shi arrangement, discussing the finite field method for determining the characteristc polynomial of $\mathcal{A}$, and using the finite field method on the Shi arrangement and hyperplane arrangements generated by Coxeter groups.

1. Introduction

Let K be a field. Let a *linear hyperplane* be an $(n-1)$-dimensional subspace of K^n of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0,$$

where the $(a_1, \ldots, a_n) \in K^n$ is the normal vector and the x_i are coordinate directions in K^n. Let a *hyperplane* be any translate of a linear hyperplane. Though hyperplanes generalize to all vector spaces of the form K^n, unless otherwise specified, we will let $K = \mathbb{R}$.

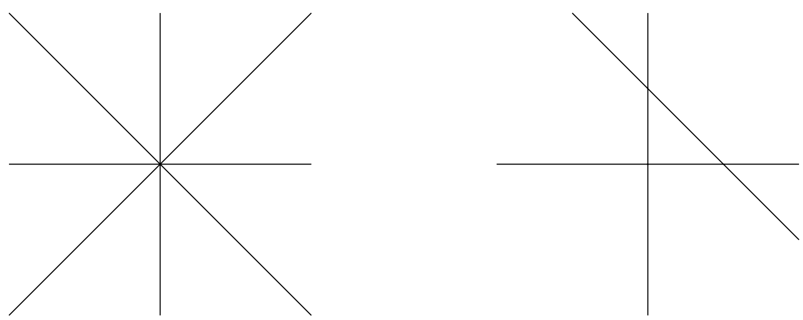

Figure 1. Examples hyperplane arrangements in $\mathbb{R}^2$.

A *hyperplane arrangement* denoted $\mathcal{A}$ is a finite set of hyperplanes. See Figure 1 for examples of hyperplane arrangements. A *region* is a connected component of

$$\mathbb{R}^n - \bigcup_{i=1}^{k} H_i,$$

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where $H_1, \dots, H_k$ are the hyperplanes in $\mathcal{A}$. A region is *bounded* if it can be contained in a sphere of some radius centered at the origin. In this paper, we will primarily consider the the set of regions formed by $\mathcal{A}$ denoted as $R(\mathcal{A})$ and the number of regions denoted as $r(\mathcal{A})$.

We now present a brief history of hyperplane arrangements. Crapo and Rota [2] formalized theory in geometric lattices and matroids which relate to the intersection posets of hyperplane arrangements. Zaslavsky [5] determined the number of regions a hyperplane arrangement divides space into using the characteristic polynomial of the arrangement. Stanley, Postnikov, Pak and others (see sections and references in [4]) determined the number of regions for different arrangements like the Shi arrangement, Linial arrangement, and Braid arrangement using bijections among other techniques. Athanasiadis [1] developed a different way of determining the number of regions in these arrangements called the finite field method.

In Section 2, we provide the preliminaries for hyperplane arrangements, namely, the intersection poset, ordered pair functions, and the characteristic polynomial. We use these tools to find a formula for the number of regions in Section 3. We also find the number of regions in the Shi arrangement through a fascinating bijections to parking functions Section 4. In Section 5, we describe the finite field method and apply it to the Shi arrangement and Catalan arrangement. Finally in Section 6, we describe the connection between hyperplane arrangements and Coxeter groups and find the number of regions in two infinite classes of such hyperplane arrangements.

2. Preliminaries

In this section, we discuss the intersection poset, ordered pair functions, and the characteristic polynomial. See [4] for the preliminaries in more depth.

But first, we must discuss some terminology for a hyperplane arrangement $\mathcal{A}$ consisting of hyperplanes $H_1, \dots, H_k$ in K^n.

- A hyperplane in $\mathcal{A}$ is *linear* if it contains the origin and *affine* otherwise.
- The *defining polynomial* of $\mathcal{A}$ is
$$Q_{\mathcal{A}} = (L_1(\mathbf{x}) - b_1) \cdots (L_k(\mathbf{x}) - b_k),$$
for $\mathbf{x} \in \mathbb{R}^n$ where the set of solutions to $L_i(\mathbf{x}) - b_i = 0$ is H_i.
- The *dimension* of $\mathcal{A}$ denoted as $\dim(\mathcal{A})$ is n, the dimension of the space $\mathcal{A}$ is in.
- The *rank* of $\mathcal{A}$ denoted as $\operatorname{rank}(\mathcal{A})$ is the dimension of the space spanned by the normals of the hyperplanes in $\mathcal{A}$.
- We say $\mathcal{A}$ is in *general position* if the intersection of p distinct hyperplanes in $\mathcal{A}$ has dimension $\max(n-p, 0)$.
- We say $\mathcal{A}$ is *central* if the intersection of the hyperplanes in $\mathcal{A}$ is nonempty.

For example the left hyperplane arrangement in Figure 1 is central while the right hyperplane arrangement is in general position.

2.1. **Intersection Poset.**

A poset or partially ordered set is a set S together with a comparison operation $\leq$ satisfying three properties

- (Identity) $x \leq x$ for all $x \in S$,
- (Antisymmetry) if $x \leq y$ and $y \leq x$ then $x = y$ for all $x, y \in S$,
- (Transitivity) if $x \leq y$ and $y \leq z$ then $x \leq z$ for all $x, y, z \in S$.

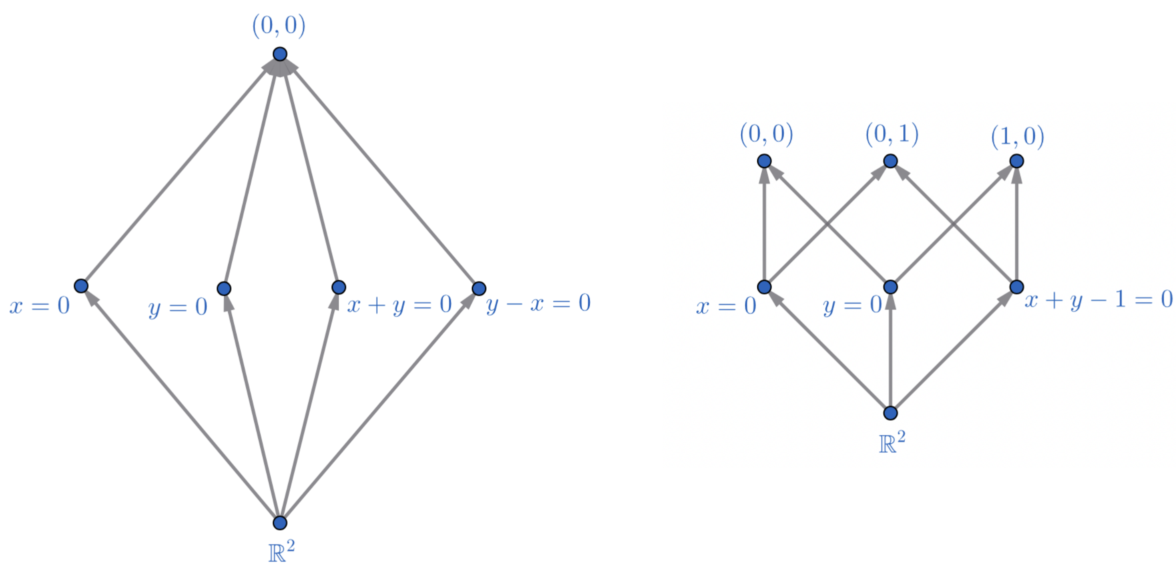

FIGURE 2. The intersection posets of the arrangements depicted in Figure 1.

Posets are often depicted in a graphical format with vertices representing elements of S and directed edges $u \to v$ meaning $u \leq v$.

The intersection poset contains all nonempty intersections of the hyperplanes in an arrangement $\mathcal{A}$. Note that the entire space $\mathbb{R}^n$ is considered an intersection of the hyperplanes (intersection of none of the hyperplanes). The intersections are ordered by reverse inclusion, i.e., if S and T are intersections of hyperplanes in $\mathcal{A}$ and $S \subseteq T$ then $T \leq S$. We let the intersection poset of $\mathcal{A}$ be denoted as $L(\mathcal{A})$.

Example 2.1. Consider the hyperplane arrangements in Figure 1. We will refer the hyperplane arrangements by their equation, e.g., the left arrangement consists of $x = 0$, $y = 0$, $y - x = 0$, and $x + y = 0$ while the right arrangement consists of $x = 0$, $y = 0$, and $x + y - 1 = 0$. Their corresponding posets are shown in Figure 2.

2.2. **Ordered Pair Functions.** Let an *ordered pair* be a tuple (x, y) for $x, y \in L(\mathcal{A})$ where $x \leq y$. Let $O(\mathcal{A})$ be the set of ordered pairs in $L(\mathcal{A})$. An *ordered pair function* is a function of the form $f : O(\mathcal{A}) \to \mathbb{Z}$. Define a convolution of two ordered pair functions f and g to be

$$f * g(x, y) = \sum_{\substack{z \in L(\mathcal{A}) \\ x \leq z \leq y}} f(x, z) g(z, y).$$

Note that the identity of the convolution operation is the function $\delta : O(\mathcal{A}) \to \mathbb{Z}$ such that

$$\delta(x, y) = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ otherwise} \end{cases}.$$

Also note that the convolution is associative as

$$(f * g) * h(x,y) = \sum_{\substack{w\in L(\mathcal{A})\\ x\le w\le y}} \left(\sum_{\substack{z\in L(\mathcal{A})\\ x\le z\le w}} f(x,z)g(z,w) \right) h(w,y) = \sum_{\substack{z,w\in L(\mathcal{A})\\ x\le z\le w\le y}} f(x,z)g(z,w)h(w,y),$$

and

$$f * (g * h)(x,y) = \sum_{\substack{z\in L(\mathcal{A})\\ x\le z\le y}} f(x,z) \left(\sum_{\substack{w\in L(\mathcal{A})\\ z\le w\le y}} g(z,w)h(w,y) \right) = \sum_{\substack{z,w\in L(\mathcal{A})\\ x\le z\le w\le y}} f(x,z)g(z,w)h(w,y).$$

Let the inverse of an ordered pair function f be a function g such that $f * g = \delta$. Because $(g * f) * g = g * (f * g) = g$, and the identity ordered pair function is unique, it also follows that $g * f = \delta$.

2.3. **Characteristic Polynomial.** Consider the ordered pair function $\mathbf{1}(x,y) = 1$. Let its inverse be the Möbius function $\mu(x,y)$. Thus, μ is defined as the unique function where $\mu(x,x) = 1$ and for $(x,y) \in O(\mathcal{A})$

$$\sum_{\substack{z\in L(\mathcal{A})\\ x\le z\le y}} \mu(x,z) = 0.$$

Example 2.2. Consider the intersection posets in Figure 2. Figure 3 depicts the intersection poset with each element $x \in L(\mathcal{A})$ labeled with $\mu(\mathbb{R}^2, x)$.

The *characteristic polynomial* of an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is the polynomial

$$\chi_{\mathcal{A}}(t) = \sum_{x\in L(\mathcal{A})} \mu(\mathbb{R}^n, x)t^{\dim(x)}.$$

Example 2.3. The characteristic polynomial of the the hyperplanes in Figure 1 are $t^2 - 4t + 3$ and $t^2 - 3t + 3$, respectively.

Let us develop an alternative way of forming the characteristic polynomial. Let $\mathcal{A}$ be a hyperplane arrangement consisting of hyperplanes $H_1, \ldots, H_k$. Let us express the space

$$X = \mathbb{R}^n - \bigcup_{i=1}^{k} H_i,$$

through set additions and subtractions of the elements of $L(\mathcal{A})$. Through the Principle of Inclusion and Exclusion, we see that

$$X = \mathbb{R}^n - \left(\sum_{1\le i\le k} H_i \right) + \left(\sum_{1\le i_1<i_2\le k} H_{i_1} \cap H_{i_2} \right) - \cdots . \tag{2.1}$$

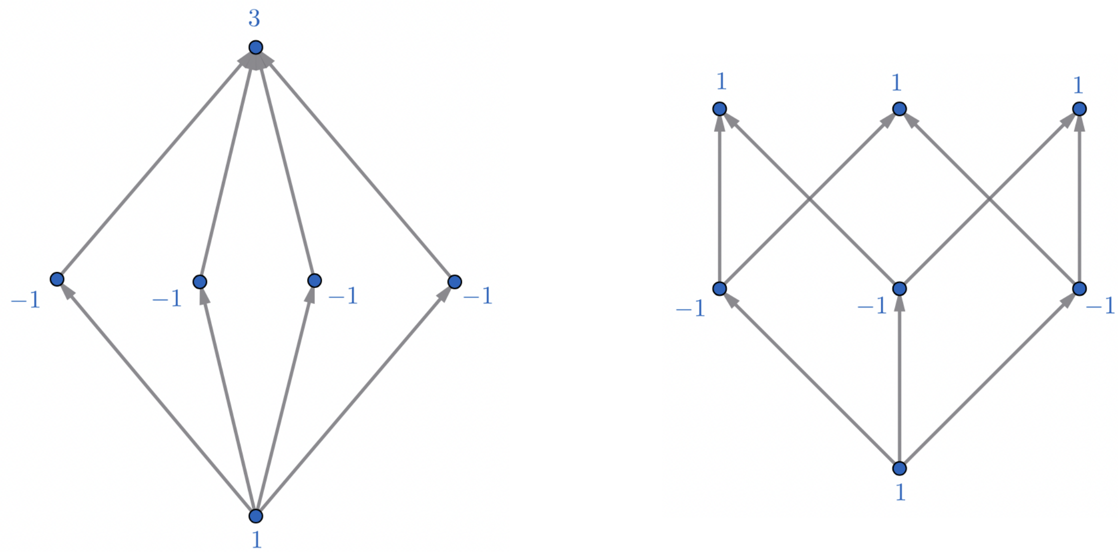

FIGURE 3. The intersection posets of the arrangements depicted in Figure 1 with Möbius labelings of the form $\mu(\mathbb{R}^2, x)$.

Now replace every nonempty intersection in Equation (2.1) including $\mathbb{R}^n$ with t^d where d is the dimension of the intersection. Let us call this polynomial $\psi_{\mathcal{A}}(t)$. In compact notation, we have

$$\psi_{\mathcal{A}}(t) = \sum_{\substack{\mathcal{B}\subseteq\mathcal{A} \\ \mathcal{B}\text{ central}}} (-1)^{|\mathcal{B}|} t^{n-\text{rank}(\mathcal{B})},$$

where $|\mathcal{B}|$ is the number of hyperplanes in $\mathcal{B}$ and $n - \text{rank}(\mathcal{B})$ refers to the dimension of the intersection of the hyperplanes in $\mathcal{B}$.

Example 2.4. To calculate $\psi_{\mathcal{A}}(t)$ for the arrangements in Figure 1, we can form the following tables. In the left arrangement, let the hyperplanes $x = 0, y = 0, y - x = 0$, and $x + y = 0$ be denoted a, b, c, and d, respectively. In the right arrangement, let the hyperplanes $x = 0, y = 0$, and $x + y - 1 = 0$ be denoted x, y, and z, respectively.

$\mathcal{B}$	$\lvert\mathcal{B}\rvert$	$\text{rank}(\mathcal{B})$
$\mathbb{R}^n$	0	0
a	1	1
b	1	1
c	1	1
d	1	1
$a \cap b$	2	2
$a \cap c$	2	2
$a \cap d$	2	2
$b \cap c$	2	2
$b \cap d$	2	2
$c \cap d$	2	2
$a \cap b \cap c$	3	2
$b \cap c \cap d$	3	2
$c \cap d \cap a$	3	2
$d \cap a \cap b$	3	2
$a \cap b \cap c \cap d$	4	2

$\mathcal{B}$	$\lvert\mathcal{B}\rvert$	$\text{rank}(\mathcal{B})$
$\mathbb{R}^n$	0	0
x	1	1
y	1	1
z	1	1
$x \cap y$	2	2
$y \cap z$	2	2
$z \cap x$	2	2

Therefore, $\chi_{\mathcal{A}}(t)$ is $t^2 - 4t + 6 - 4 + 1 = t^2 - 4t + 3$ in the left arrangement and $t^2 - 3t + 3$ in the right arrangement.

Theorem 2.5. *For all hyperplane arrangements $\mathcal{A}$ in $\mathbb{R}^n$, we have $\chi_{\mathcal{A}}(t) = \psi_{\mathcal{A}}(t)$.*

Proof. Let us convert $\chi_{\mathcal{A}}(t)$ into an addition and subtraction of sets as in Equation (2.1)

$$Y = \sum_{x \in L(\mathcal{A})} \mu(\mathbb{R}^n, x)x.$$

We will now show that the coefficient of the intersections in X and Y are the same through induction. The coefficient of $x = \mathbb{R}^n$ are both 1 as $\mathbb{R}^n$ appears with coefficient 1 exactly once in X and $\mu(\mathbb{R}^n, \mathbb{R}^n) = 1$ is the coefficient of $\mathbb{R}^n$ in Y.

Suppose we want to show the coefficient of x in X and Y is the same now and we already know the coefficient of all $y \leq x$ is the same. Let $[z]X$ and $[z]Y$ refer to the coefficient of the intersection z in X and Y respectively. In Y, we see that

$$[x]Y = -\sum_{\substack{y \in L(\mathcal{A}) \\ y < x}} \mu(\mathbb{R}^n, y) = -\sum_{\substack{y \in L(\mathcal{A}) \\ y < x}} [y]Y.$$

In X, we know if $y \leq x$, then y contains x. There must be a point $\mathbf{x} \in x$ such that $\mathbf{x}$ is contained in all $y \leq x$ but in no other intersections. Define the count of $\mathbf{x}$ to be the number of times it is added and subtracted in the sum X, i.e.,

$$\sum_{\substack{y \in L(\mathcal{A}) \\ y \leq x}} [y]X.$$

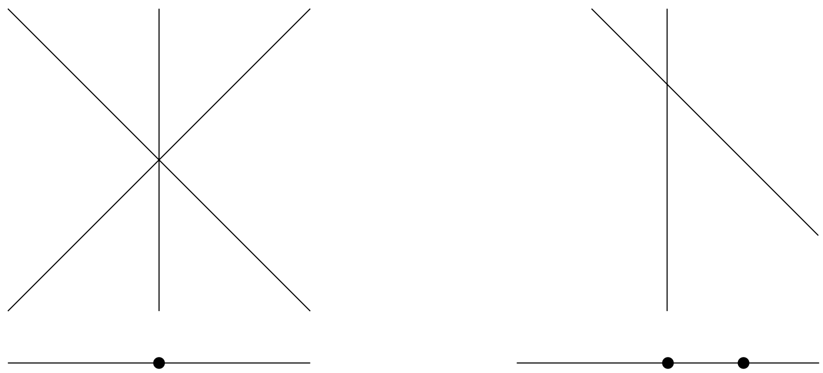

FIGURE 4. The deletion and restriction of the arrangements in Figure 1 with respect to the hyperplane $x = 0$. The deletion and restriction for the left arrangement are shown on the left above and the deletion and restriction for the right arrangement are shown on the right above.

In order for X to not include $\mathbf{x}$, the count of $\mathbf{x}$ must be 0. Therefore,

$$[x]X = -\sum_{\substack{y \in L(\mathcal{A}) \\ y < x}} [y]X.$$

It follows that $[x]X = [x]Y$, so $\chi_{\mathcal{A}}(t) = \psi_{\mathcal{A}}(t)$. ■

3. Number of Regions

Let $R(\mathcal{A})$ and $r(\mathcal{A})$ refer to the set and number of regions for a hyperplane arrangement $\mathcal{A}$, respectively. We will find a relationship between $r(\mathcal{A})$ and $\chi_{\mathcal{A}}(t)$. To do this, we will need a technique called deletion-restriction. This proof was originally developed by Zaslavsky [5]. See [4] for more detail.

Let H be a hyperplane in $\mathcal{A}$. Define $\mathcal{A}_H$ or the *deletion of $\mathcal{A}$ with respect to H* to be the hyperplane arrangement $\mathcal{A} \setminus H$. Define $\mathcal{A}^H$ or the *restriction of $\mathcal{A}$ with respect to H* to be the hyperplane arrangement $\{H' \cap H : H' \in \mathcal{A}\}$ where the hyperplane arrangement exists in the subspace H. Note that the subspace H is isomorphic to $\mathbb{R}^{n-1}$ and can be treated as such. Therefore, the characteristic polynomial of $\mathcal{A}_H$ still exists.

Example 3.1. The deletion and restriction of the arrangements in Figure 1 with respect to the hyperplane $x = 0$ are shown in Figure 4.

Proposition 3.2. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$ and H a hyperplane within $\mathcal{A}$. Then*

$$r(\mathcal{A}) = r(\mathcal{A}_H) + r(\mathcal{A}^H).$$

Proof. Consider any region T in $R(\mathcal{A}_H)$. Notice that T must be convex as it is the intersection of convex half-spaces. It follows that H can be cut T into at most two pieces. Therefore, $r(\mathcal{A})$ is $r(\mathcal{A}_H)$ plus the number of regions in $R(\mathcal{A}_H)$ which are cut into two pieces by H. Let $S \subseteq R(\mathcal{A}_H)$ be the set of these regions. Notice $\{R \cap H : R \in S\} = R(\mathcal{A}^H)$. Therefore, we have $|S| = |R(\mathcal{A}^H)| = r(\mathcal{A}^H)$ as every element in S has a corresponding element in $R(\mathcal{A}^H)$, and vice versa. Thus,

$$r(\mathcal{A}) = r(\mathcal{A}_H) + r(\mathcal{A}^H).$$

■

Example 3.3. For the arrangements in Figure 1, letting H be the hyperplane $x = 0$, we see

$$8 = r(\mathcal{A}) = r(\mathcal{A}_H) + r(\mathcal{A}^H) = 6 + 2,$$

for the left arrangement $\mathcal{A}$ and

$$7 = r(\mathcal{B}) = r(\mathcal{B}_H) + r(\mathcal{B}^H) = 4 + 3,$$

for the right arrangement $\mathcal{B}$.

We have a similar equation for $\chi_{\mathcal{A}}(t)$.

Proposition 3.4. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$ and H a hyperplane within $\mathcal{A}$. Then*

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}_H}(t) - \chi_{\mathcal{A}^H}(t).$$

Proof. Recall the definition of $\psi_{\mathcal{A}}(t) = \chi_{\mathcal{A}}(t)$. Let X, Y, and Z be the complement of the hyperplanes in $\mathcal{A}$, $\mathcal{A}_H$, and $\mathcal{A}^H$, respectively. We know $X = Y - Z$. If we write X, Y, and Z into the form of Equation (2.1), we get two equivalent expressions for the complement of the hyperplanes in $\mathcal{A}$ in terms of intersections of the hyperplanes within $\mathcal{A}$. Note that the coefficient of any intersection must be the same must be the same in the two expressions. Replacing every intersection x with $t^{\dim(x)}$, we see

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}_H}(t) - \chi_{\mathcal{A}^H}(t).$$

■

Example 3.5. For the arrangements in Figure 4, letting H be the hyperplane $x = 0$, we see

$$t^2 - 4t + 3 = \chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}_H}(t) - \chi_{\mathcal{A}^H}(t) = (t^2 - 3t + 2) - (t - 1),$$

for the left arrangement $\mathcal{A}$ and

$$t^2 - 3t + 3 = \chi_{\mathcal{B}}(t) = \chi_{\mathcal{B}_H}(t) - \chi_{\mathcal{B}^H}(t) = (t^2 - 2t + 1) - (t - 2),$$

for the right arrangement $\mathcal{B}$.

We can now illustrate the relationship between $r(\mathcal{A})$ and $\chi_{\mathcal{A}}(t)$.

Theorem 3.6. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. Then*

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1).$$

Proof. Say $\mathcal{A}$ contains no hyperplanes. Then $r(\mathcal{A})$ is 1. We also have $\chi_{\mathcal{A}}(t) = t^n$, so $(-1)^n \chi_{\mathcal{A}}(-1) = 1$ as well.

Now consider a any arrangement $\mathcal{A}$. Assume that all hyperplane arrangements $\mathcal{B}$ with fewer hyperplanes than $\mathcal{A}$ or with lower dimension satisfy $r(\mathcal{B}) = (-1)^n \chi_{\mathcal{B}}(-1)$. Therefore,

$$r(\mathcal{A}) = r(\mathcal{A}_H) + r(\mathcal{A}^H) = (-1)^n \chi_{\mathcal{A}_H}(-1) + (-1)^{n-1} \chi_{\mathcal{A}^H}(-1) = (-1)^n \chi_{\mathcal{A}}(-1).$$

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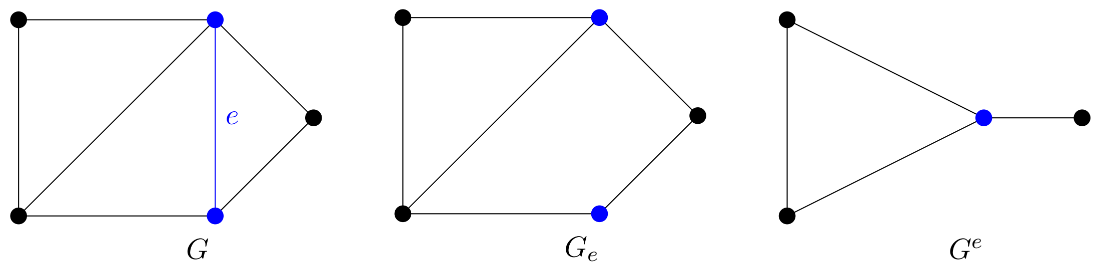


FIGURE 5. Deletion and contraction of G with respect to e.

Example 3.7. In the hyperplane arrangements in Figure 1, we see that the left hyperplane arrangement $\mathcal{A}$ has 8 regions which is $(-1)^2\chi_{\mathcal{A}}(-1) = (-1)^2 - 4(-1) + 3 = 8$ and the right hyperplane arrangement $\mathcal{B}$ has 7 regions which is $(-1)^2\chi_{\mathcal{B}}(-1) = (-1)^2 - 3(-1) + 3 = 7$.

3.1. **Acyclic Orientations.** We now introduce a connection between the deletion-restriction in the characteristic polynomial and the deletion-contraction in the chromatic function for graphs. We closely follow [4].

Let us start by defining the chromatic function in graphs. A proper coloring of a graph G on k colors is a function $\kappa : V(G) \to \{1, \ldots, k\}$ such that $\kappa(u) \neq \kappa(v)$ if $(u, v) \in E(G)$. The *chromatic function* denoted $\chi_G : \mathbb{Z}^+ \to \mathbb{N}$ is defined by $\chi_G(t)$ is the number of proper colorings with t colors.

The chromatic function follows a similar relation to deletion-restriction for the characteristic polynomial. The relation is called *deletion-contraction*. Let G be a graph on n vertices and e be an edge in G connecting vertices u and v. Define G_e or the *deletion* of G with respect to e to be the graph with e removed. Define G^e or the *contraction* of G with respect to e to be the graph where u and v become superimposed into one vertex x such that all edges with an endpoint at u and v now have an endpoint at x. Remove any duplicated edges, i.e., if w shares an edge with u and v, we will get two copies of (w, x) if we do this process, so we remove one of these copies. See Figure 5 for an example of the deletion and contraction of a graph.

Proposition 3.8 (Deletion-Contraction)**.** *For a graph G and an edge $e = (u, v)$ in G,*

$$\chi_G(t) = \chi_{G_e}(t) - \chi_{G^e}(t).$$

Proof. We can think of the equation instead as $\chi_{G_e}(t) = \chi_G(t) + \chi_{G^e}(t)$. Let κ be a proper coloring of G_e. If $\kappa(u) \neq \kappa(v)$, then we can add the edge e and κ would still be a proper coloring of G. If $\kappa(u) = \kappa(v)$ on the other hand, then we can combine them into one vertex through a contraction and the result would be a proper coloring of G^e. Every coloring κ falls into one of these categories and every coloring of G or G^e can be converted into a coloring of G_e. Therefore,

$$\chi_{G_e}(t) = \chi_G(t) + \chi_{G^e}(t).$$

■

Remark 3.9. Because of this relation, we can show that $\chi_G(t)$ is a polynomial. When G is a graph on n vertices with no edges, $\chi_G(t) = t^n$, and through deletion contraction, we can always express $\chi_G(t)$ as a sum of characteristic functions of graphs with no edges.

We now illustrate the relationship between hyperplane arrangements and graphs. Let G be a graph on n vertices. Label its vertices $x_1, \dots, x_n$. Let $\mathcal{A}_G = \{x_i - x_j = 0 : (x_i, x_j) \in E(G)\}$.

Proposition 3.10. *For a graph G on n vertices,*

$$\chi_{\mathcal{A}_G}(t) = \chi_G(t).$$

Proof. We can use the fact that both the characteristic polynomial and chromatic function follow a deletion-contraction/restriction type relation. Let two arrangements $\mathcal{A}$ and $\mathcal{B}$ satisfy $\mathcal{A} \cong \mathcal{B}$ if there exists a linear transformation from the vector space containing $\mathcal{A}$ to vector space containing $\mathcal{B}$ such that the points in $\mathcal{A}$ are mapped to the points of $\mathcal{B}$ and vice versa. If $\mathcal{A} \cong \mathcal{B}$, then $\chi_{\mathcal{A}}(t) = \chi_{\mathcal{B}}(t)$. In order to show that the deletion-contraction in graphs and the deletion-restriction in hyperplane arrangements are the same, we must verify that $\mathcal{A}_{G_e} \cong (\mathcal{A}_G)_H$ and $\mathcal{A}_{G^e} \cong (\mathcal{A}_G)^H$ where e is the edge (x_i, x_j) and H is the hyperplane $x_i - x_j = 0$ to make sure the relations are completely analogous.

The equation $\mathcal{A}_{G_e} \cong (\mathcal{A}_G)_H$ is true as when we remove e form the graph, we remove $x_i - x_j = 0$ from $\mathcal{A}_G$. For the second equation, start with $(\mathcal{A}_G)^H$. Consider the map $\phi : H \to \mathbb{R}^{n-1}$ where

$$\phi(a_1x_1 + \cdots + ax_i + \cdots + ax_j + \cdots + a_nx_n) = a_1x_1 + \cdots + ax_i + \cdots + 0x_j + \cdots + a_nx_n,$$

where there is no x_j term. Consider a hyperplane $x_p - x_q = 0$ in $(\mathcal{A}_G)^H$. Through ϕ, if neither p nor q are i or j, then $x_p - x_q = 0$ is mapped to $x_p - x_q = 0$ which is a hyperplane in $\mathcal{A}_{G^e}$ while if one of p and q, say p is i or j, then $x_p - x_q = 0$ is mapped to $x_i - x_q = 0$ which is a hyperplane in $\mathcal{A}_{G^e}$. All hyperplanes in $\mathcal{A}_{G^e}$ can be achieved in this way, so $\mathcal{A}_{G^e} \cong (\mathcal{A}_G)^H$.

When G has no edges, $\chi_G(t) = t^n$ and $\mathcal{A}_G$ has no hyperplanes so $\chi_{A_G}(t) = t_n$. Assume that G is some graphs and it is shown that for all graphs with fewer vertices or less edges than G the characteristic polynomial and chromatic function are equal. Using deletion-contraction and deletion-restriction on some edge e in G and its corresponding hyperplane H, we then have

$$\chi_{A_G}(t) = \chi_{(A_G)_H}(t) - \chi_{(A_G)^H}(t) = \chi_{A_{G_e}}(t) - \chi_{A_{G^e}}(t) = \chi_{G_e}(t) - \chi_{G^e}(t) = \chi_G(t).$$

Thus, through induction $\chi_{A_G}(t) = \chi_G(t)$. ■

Define an *orientation* $\mathfrak{o}$ of a graph G to be a directed graph G' with the same vertex set and edge set as G except the edges are no directed. An orientation is *acyclic* if there are no directed cycles. We will now illustrate a relationship between the regions of $\mathcal{A}_G$ and the acyclic orientations of G.

Theorem 3.11. *Let G be a graph on n vertices. There is a bijection between the acyclic orientations of G and the regions of $\mathcal{A}_G$.*

Proof. Consider a region S of $\mathcal{A}_G$. We can express S as the intersection of inequalities of the form $x_i > x_j$ where $(i, j) \in E(G)$. Notice that because S is nonempty it is impossible for there to be $i_1, i_2, \dots, i_k$ such that

$$x_{i_1} > x_{i_2} > \cdots > x_{i_k} > x_{i_1},$$

as that would imply there is no solution. With these inequalities, we can construct G' with vertex set $\{x_1, \dots, x_n\}$ where there is a directed edge from x_j to x_i if in S, we have $x_i > x_j$. Note that for all $(i, j) \in E(G)$, the region S must lie on one side of $x_i = x_j$, so all edges in G will appear as some directed edge in G'. Let this function mapping regions to acyclic orientations be denoted f, i.e., $f(S) = G'$ in the manner described above.

We now show that f is a bijection. First we show injectivity. Consider distinct regions S and T. Because they are distinct, there must be some hyperplane in $\mathcal{A}_G$ say $x_i - x_j = 0$, where S and T lie on opposite sides of it. It follows that in $f(S)$ and $f(T)$ the edge between x_i and x_j are pointing in opposite directions.

We now show surjectivity. Consider some acyclic orientation G' of G. We will prove that there exists some labeling $L : V(G') \to \{1, \dots, n\}$ of the vertices of G' with numbers in $\{1, \dots, n\}$ such that if there exists a directed edge from x_i to x_j, then the label for x_i is less than the label for x_j. We will induct on the number of vertices. In all one vertex acyclic orientations, the one vertex can be labeled 1 and we are done. Assume that all $n-1$ acyclic orientations have a labeling with $\{1, \dots, n-1\}$. Now consider the acyclic orientation G' on n vertices. Notice that G' must have some *sink vertex* v, i.e., v has no directed edges out of it, as if we start at a random vertex u and continue along a directed path from u, the path must end at a sink vertex as there are a finite number of vertices and no cycles. Label v with n and remove it to get an acyclic orientation with $n-1$ vertices and possibly multiple connected components. We can label this graph with $\{1, \dots, n-1\}$ by the inductive hypothesis, so we now have the labeling L for G'. Let

$$\mathbf{x} = (L(x_1), \dots, L(x_n)).$$

Consider the region S containing $\mathbf{x}$. This region has the acyclic orientation G'. Thus f is surjective. ■

This bijection between regions and acyclic orientations also implies the following fascinating result.

Corollary 3.12. *The number of acyclic orientations of a graph G on n vertices is* $(-1)^n \chi_G(-1)$.

Proof. The number of regions of $\mathcal{A}_G$ is $(-1)^n \chi_{\mathcal{A}_G}(-1)$. Because of the bijection, the number of acyclic orientations of G is

$$(-1)^n \chi_{\mathcal{A}_G}(-1) = (-1)^n \chi_G(-1).$$

■

4. Shi Arrangement and Parking Functions

Define the Shi Arrangement $\mathcal{S}_n$ to consist of the hyperplanes

$$\{x_i - x_j = 0 : 1 \le i < j \le n\} \cup \{x_i - x_j = 1 : 1 \le i < j \le n\}.$$

This section is about determining the number of regions formed by the arrangement. See [4] for more detail.

Ultimately, we will establish a bijection between the regions of the Shi arrangement and parking functions, but we must first define parking functions. A *parking function* is a sequence of integers $(a_1, \dots, a_n)$ such that its sorted sequence $(b_1, \dots, b_n)$ satisfies $1 \le b_i \le i$. Parking functions arise from the following problem.

Let there be n cars numbered 1 to n queued to park in n parking spots numbered 0 to $n-1$. Car i has a preferred parking spot a_i. One by one the cars enter the parking lot. If their preferred

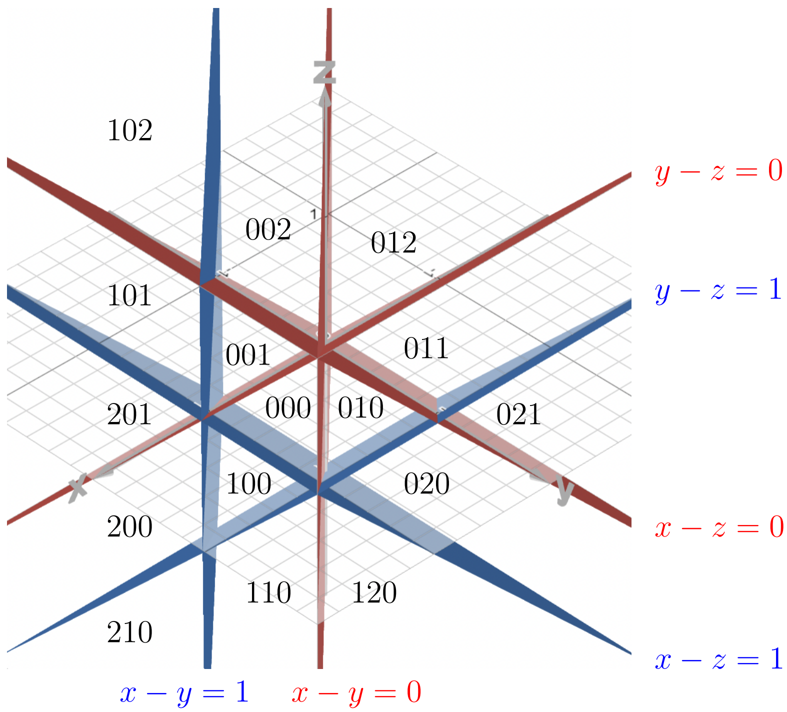


FIGURE 6. The Shi arrangement $\mathcal{S}_3$. In the diagram x_1, x_2, x_3 are replaced with The red hyperplanes are the planes of the form $x_i - x_j = 0$ while the blue hyperplanes are of the form $x_i - x_j = 1$. The labels in the regions are parking functions.

parking spot is available, they park there. Otherwise, they park at the first available parking spot after their preferred parking spot.

Proposition 4.1. *The cars can park if and only if their preferred parking spots* $(a_1, \dots, a_n)$ *form a parking function.*

Proof. Assume that $(a_1, \dots, a_n)$ is a parking function. Consider some point in time in the parking process and let k be last open parking spot. This would imply that there are $n-k$ cars with a preferred parking spot larger than k. Because $(a_1, \dots, a_n)$ is a parking function, there can be no other cars with a preferred parking spot larger than k. Thus the next car can always park.

We can modify the logic above to prove the other direction. Assume $(a_1, \dots, a_n)$ is not a parking function. Then there is some k for which there are more than $n-k$ cars with a preferred parking spot larger than k. Therefore even if $n-k$ of the cars manage to park, they will fill all parking spots with number larger than k, so the next car with a preferred parking spot larger than k cannot park. ∎

We will now define a bijection $\lambda : R(\mathcal{S}_n) \to \mathbb{Z}^n$ from the regions of $\mathcal{S}_n$ to parking functions. Let R_0 denote the region where $x_i > x_j$ and $x_i < x_j + 1$ for all $1 \le i < j \le n$. This is the region marked 000 in Figure 6. Set $\lambda(R_0) = (0, \dots, 0)$. Let two regions S and T be separated by a hyperplane H if S and T lie on opposite sides of H. The bijection is defined for the rest of the regions as follows.

- If regions R and R' are separated by exactly one hyperplane $x_i - x_j = 0$ with $1 \leq i < j \leq n$ and R and R_0 are on the same side of the hyperplane, then $\lambda(R') = \lambda(R) + \mathbf{x}_j$ where $\mathbf{x}_j$ is the vector with 0 is all components except for a 1 in the jth position.
- If regions R and R' are separated by exactly one hyperplane $x_i - x_j = 1$ with $1 \leq i < j \leq n$ and R and R_0 are on the same side of the hyperplane, then $\lambda(R') = \lambda(R) + \mathbf{x}_i$.

Note that λ is well-defined as $\lambda(R)$ only depends on the set of hyperplanes separating R from R_0. The values for λ are illustrated in Figure 6 in the case of $\mathcal{S}_3$.

We now have to prove that λ is indeed a bijection from $R(\mathcal{S}_n)$ to the parking functions. It will help to think about every region R of $\mathcal{S}_n$ in two stages. First, we consider which region of the arrangement $\mathcal{B}_n = \{x_i - x_j = 0 : 1 \leq i < j \leq n\}$ the region R is and then we think about which region of the arrangement $\mathcal{C}_n = \{x_i - x_j = 1 : 1 \leq i < j \leq n\}$ the region R is in. The intersection of these regions will be R.

Let us first examine the regions of $\mathcal{B}_n$. This arrangement is actually known as the *Braid arrangement.* Notice that each region is the intersection of half-spaces of the form $x_i > x_j$. As a result, every region can be expressed in the form

$$x_{\sigma(1)} > x_{\sigma(2)} > \cdots > x_{\sigma(n)},$$

where σ is a permutation of $\{1, \ldots, n\}$. It follows that there is a bijection from the regions of $\mathcal{B}_n$ and permutations.

Let us determine which regions of $\mathcal{C}_n$ which intersect $x_{\sigma(1)} > x_{\sigma(2)} > \cdots > x_{\sigma(n)}$. For $i < j$ and $\sigma(i) > \sigma(j)$, we already know $x_{\sigma(i)} > x_{\sigma(j)}$, so we already know R is in the half-space $x_{\sigma(j)} < x_{\sigma(i)} + 1$. Therefore, it suffices to consider the half-spaces on the sides of the hyperplanes $x_{\sigma(i)} - x_{\sigma(j)} = 1$ where $i < j$ and $\sigma(i) < \sigma(j)$.

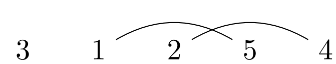
Consider if $x_{\sigma(i)} < x_{\sigma(j)} + 1$. Therefore we have the chain

$$x_{\sigma(j)} + 1 > x_{\sigma(i)} > x_{\sigma(i+1)} > \cdots > x_{\sigma(j)}.$$

From the chain, we also see $x_{\sigma(k)} < x_{\sigma(l)} + 1$ is implied for all $i \leq k < l \leq j$. Intuitively, this means that the gap between $x_{\sigma(i)}$ and $x_{\sigma(j)}$ is small and hence there is small gap between $x_{\sigma(k)}$ and $x_{\sigma(l)}$ if k and l are between i and j. Therefore, if we know $x_{\sigma(i)} < x_{\sigma(j)} + 1$, we can omit $x_{\sigma(k)} < x_{\sigma(l)} + 1$. Now consider if $x_{\sigma(i)} > x_{\sigma(j)} + 1$. Intuitively, this means that the gap between $x_{\sigma(i)}$ and $x_{\sigma(j)}$ is large. Notice then that is impossible for there to be k and l such that $k \leq i < j \leq l$ such that $x_{\sigma(k)} < x_{\sigma(l)} + 1$.

We can form a visualization of these inequalities through arcs over a permutation σ.

(4.1)



The sequence of numbers represents the inequalities defining the region in $\mathcal{B}_n$. The arcs represent the chains of inequalities, e.g. the arc between 1 and 5 shows $x_5 + 1 > x_1 > x_2 > x_5$. If $i < j$ and $\sigma(i) < \sigma(j)$ and there is no arc over both i and j, then $x_{\sigma(i)} < x_{\sigma(j)} + 1$ is impossible so we have $x_{\sigma(i)} > x_{\sigma(j)} + 1$. In the example above, the inequalities defining the region, not mentioning too many superfluous ones, would be

$$x_3 > x_1 > x_2 > x_5 > x_4,$$
$$x_1 < x_5 + 1, x_2 < x_4 + 1,$$
$$x_3 > x_1 + 1, x_1 > x_4 + 1.$$

We can split λ into two parts. Let λ_1 be defined so that $\lambda_1(R_0) = (0, \dots, 0)$ and if regions R and R' are separated by exactly one hyperplane of $\mathcal{B}_n$ and R and R_0 are on the same side of the hyperplane, then $\lambda(R') = \lambda(R) + \mathbf{x}_j$. When crossing over a hyperplane of $\mathcal{C}_n$ let λ_1 not change. Similarly define λ_2 where it does not change when crossing a hyperplane of $\mathcal{B}_n$ but does change when crossing a hyperplane of $\mathcal{C}_n$ so that $\lambda_1(R) + \lambda_2(R) = \lambda(R)$.

Note that the ith component of $\lambda_1(R)$ denoted as $\lambda_1^{(i)}(R)$ is the number of $j < i$ such that the hyperplane $x_j - x_i = 0$ is in between R and R_0. Before the hyperplane is crossed, we have $x_j > x_i$ and after it is crossed, we have $x_i > x_j$ even though $j < i$. Therefore, the number of such j is the number of j less than i appearing after x_i in the permutation σ. In other words, we have $\lambda_1^{(i)}(R)$ is number of j where $\sigma^{-1}(i) < \sigma^{-1}(j)$ and $i > j$.

The ith component of $\lambda_2(R)$ denoted as $\lambda_2^{(i)}(R)$ is the number of $j > i$ such that the hyperplane $x_i - x_j = 1$ is in between R and R_0. Before the hyperplane is crossed, we have $x_i - x_j < 1$ and after, we have $x_i - x_j > 1$. This corresponds to an absence of an arc over both i and j. Also note that $x_i - x_j > 1 > 0$, so i must appear before j in σ. Thus, we have $\lambda_2^{(i)}(R)$ is the number of j where $\sigma^{-1}(i) < \sigma^{-1}(j)$ and $i < j$, but there is no arc containing i and j.

Note that if a j is counted in $\lambda_1^{(i)}(R)$, it cannot be counted in the $\lambda_2^{(i)}(R)$ as λ_1 forces $i > j$ while λ_2 forces $i < j$. It then follows that the ith component of $\lambda(R)$ denoted $\lambda^{(i)}(R)$ is at most $n - \sigma^{-1}(i)$ as only the j after i in σ can possibly be counted in λ_1 or λ_2. Therefore, $\lambda(R)$ is always a parking function.

Example 4.2. We can illustrate this argument with the region corresponding to the permutations and arcs in (4.1). Let us first determine $\lambda_1(R)$.

(1) $\lambda_1^{(1)}$ is the number of $j < 1$ after 1 which is 0.
(2) $\lambda_1^{(2)}$ is the number of $j < 2$ after 2 which is 0.
(3) $\lambda_1^{(3)}$ is the number of $j < 3$ after 3 which is 2.
(4) $\lambda_1^{(4)}$ is the number of $j < 4$ after 4 which is 0.
(5) $\lambda_1^{(5)}$ is the number of $j < 5$ after 5 which is 1.

Let us now determine $\lambda_2(R)$.

(1) $\lambda_2^{(1)}$ is the number of $j > 1$ after 1 where there is no arc over both 1 and j which is 1.
(2) $\lambda_2^{(2)}$ is the number of $j > 2$ after 2 where there is no arc over both 2 and j which is 0.
(3) $\lambda_2^{(3)}$ is the number of $j > 3$ after 3 where there is no arc over both 3 and j which is 2.
(4) $\lambda_2^{(4)}$ is the number of $j > 4$ after 4 where there is no arc over both 4 and j which is 0.
(5) $\lambda_2^{(5)}$ is the number of $j > 5$ after 5 where there is no arc over both 5 and j which is 0.

Therefore, $\lambda(R) = (1,0,4,0,1)$. From the process, we can see that $\lambda^{(i)}(R) \leq n - \sigma^{-1}(i)$ as only the j after i could be counted towards $\lambda^{(i)}(R)$.

We must now show that $\lambda(R)$ is injective and surjective. To do this, we will look at the inverse of λ mapping parking functions to regions. We must first define terminology a prove a lemma to do this. If j is counted in the ith component of λ_1, i.e., $\sigma^{-1}(i) < \sigma^{-1}(j)$ and $i > j$, then we refer to (i,j) as a *type 1 pair.* If j is counted in the ith component of λ_2, i.e., $\sigma^{-1}(i) < \sigma^{-1}(j)$ and $i < j$ and there is no arc over i and j, then we refer to (i,j) as a *type 2 pair.*

Lemma 4.3. *If (i,j) is a type 1 or 2 pair, then $\lambda^{(i)}(R) > \lambda^{(j)}(R)$*

Proof. Suppose (i,j) is a type 1 pair. Then, j is smaller than i but after i in the permutation σ. It follows that $\lambda_1^{(i)}(R) > \lambda_1^{(j)}(R)$, as k counted in $\lambda_1^{(j)}(R)$ is smaller than $j > i$ and further past the $\sigma^{-1}(j) > \sigma^{-1}(i)$ position, so k must be counted in $\lambda_1^{(i)}(R)$ as well. As for $\lambda_2(R)$, any $k > i$ after j in the permutation σ with no arc over j and k must be counted in both $\lambda_2^{(i)}(R)$ and $\lambda_2^{(j)}(R)$. Thus, we are left with the k satisfying $j < k < i$ after j in σ with no arc over both j and k. But notice then that even though k is counted in $\lambda_2^{(j)}(R)$ but not $\lambda_2^{(i)}(R)$, we have that k is counted in $\lambda_1^{(i)}(R)$. as k is smaller than i but after i in σ. Therefore, $\lambda^{(i)}(R) > \lambda^{(j)}(R)$.

Now suppose (i,j) is a type 2 pair. Then j is larger than i and after i in σ but there is no arc over both i and j. Notice then that all k after j and larger than j with no arc over k and j are counted in $\lambda_2^{(j)}(R)$ and $\lambda_2^{(i)}(R)$ as k is larger than i and after i and there is no arc over k and i. Similarly, all k after j and smaller than i are counted in both $\lambda_1^{(i)}(R)$ and $\lambda_1^{(j)}(R)$. Finally, all k after j and satisfying $i < k < j$ are counted in $\lambda_1^{(j)}(R)$ and $\lambda_2^{(i)}(R)$ as k is larger than i and there cannot be an arc over both i and k as there is no arc over both i and j. ■

Example 4.4. In the permutations and arcs of (4.1), Lemma 4.3 implies that

$$\lambda^{(1)} > \lambda^{(4)}, \lambda^{(3)} > \lambda^{(1)}, \lambda^{(3)} > \lambda^{(2)}, \lambda^{(3)} > \lambda^{(5)}, \lambda^{(3)} > \lambda^{(4)}, \lambda^{(5)} > \lambda^{(4)}.$$

With Lemma 4.3, we can now develop a method of finding the inverse of λ. We will do this by constructing the permutations and arcs sequentially, i.e., starting with one number, then adding another number and its arcs, then adding another number and its arcs, and so on. Start with the parking function α. Let $1 \leq i \leq n$ be some integer. The goal when constructing the permutations and arcs sequentially is that when we add the next number, we can determine is position and the arcs over it exactly. Consider iterating through $\{1,\ldots,n\}$ starting with the i with the smallest $\alpha(i)$ from largest to smallest and then moving onto the i with the next smallest $\alpha(i)$ moving from largest to smallest and so on. In Equation (4.1), we know $\lambda = (1,0,4,0,1)$, so we would traverse $\{1,\ldots,5\}$ in the order $4,2,5,1,3$. Let this order be represented by the permutation τ. If we use this order, whenever we add $\tau(i)$, the only possible j which could form a type 1 or type 2 pair $(\tau(i),j)$ are the j satisfying $\alpha(j) < \alpha(\tau(i))$ by Lemma 4.3, so $\tau^{-1}(j) < i$. Thus, j is already in the permutation and arcs, so we can just place $\tau(i)$ in the location where the number of type 1 or 2 pairs $\tau(i)$ forms is $\alpha(\tau(i))$.

We must show that there exists a unique position and arcs satisfying this. To do this, let us first determine where $\tau(i)$ can be added. In order to not affect $\lambda^{(j)}(R)$ for any other j already in the permutation, $\tau(i)$ must be placed before the first element larger than $\tau(i)$ in the permutation.

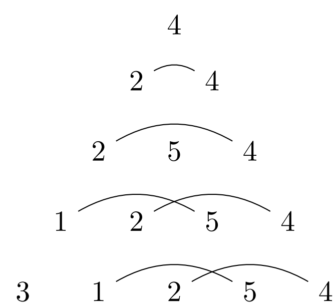
In addition, no arc can be added which does not start or end at $\tau(i)$ as this will necessarily affect $\lambda^{(j)}(R)$ where j is is the number at the start of the arc. It follows that at most two arcs can appear with one start at $\tau(i)$ and the other ending at $\tau(i)$.

We sketch a proof of the existence of such placement and arcs. We do this through induction by starting with $\lambda^{(\tau(i))}(R)$ large and decreasing $\lambda^{(\tau(i)}(R)$ by 1 at each step if possible. Start by placing $\tau(i)$ in the leftmost location in the permutation. Let there be no arc over $\tau(i)$. Thus, we have $\lambda^{(\tau(i))}(R) = i - 1$. This is the maximum possible value of $\lambda^{(\tau(i))}$ because $\alpha + \mathbf{1}$ is a parking function.

Assume that $\tau(i)$ is in the pth position. Let the elements before it be $j_1, \dots, j_{p-1}$ and the elements after it be $j_{p+1}, \dots, j_i$. Let there be an arc from j_e to $\tau(i)$ and an arc from $\tau(i)$ to j_f for $1 \leq e \leq p$ and $p \leq f \leq i$. Let these arcs be referred to as the *left arc* and *right arc* respectively. If the left arc or right arc don't actually exist, let $e = p$ or $f = p$, respectively. Let these arcs be *redundant* meaning if $\tau(i)$ is within an arc from $j_{e'}$ to $j_{f'}$ then $e \leq e'$ and $f \geq f'$ but it is possible for the left arc or right arc to be contained in another arc.

We now take cases on whether $\tau(i) < j_{p+1}$. Depending on this, we either swap $\tau(i)$ and j_{p+1} causing us to possible change e or we increase f. If we cannot do either, it can be proved that we always find an element j_y such that $\lambda^{(j_y)}(R) \geq \lambda^{(\tau(i))}(R)$. Because τ is ordered by α, it is not necessary to decrease $\lambda^{(\tau(i))}(R)$. The conditions on these two changes are complicated though, so we provide the following example instead

Example 4.5. We can illustrate the sequence of permutations and arcs in the case of (4.1).



- We start by adding $\tau(1) = 4$ to the permutation.
- We add $\tau(2) = 2$ to the permutation. We know 2 must lie before 4 so $\lambda^{(4)}(R)$ does not increase. Because $\lambda^{(2)}(R) = 0$, there must be an arc over 2 and 4.
- We add $\tau(3) = 5$. In order for $\lambda^{(5)}(R) = 1$, 5 must lie behind exactly one element in σ.
- We add $\tau(4) = 1$. In order to not increase $\lambda^{(2)}(R)$, $\lambda^{(5)}(R)$, or $\lambda^{(4)}(R)$, we see 1 must lie before $2, 5$, and 4 in σ. We have $\lambda^{(1)}(R) = 1$, so there must be an arc between 1 and 5.
- We add $\tau(5) = 3$. In order for $\lambda^{(3)} = 4$, we see 3 must lie at the start of σ. There cannot be any arcs over 3.

As for uniqueness of the position, we first make two observations

(1) If we increase p, leaving f the same, then $\lambda^{(\tau(i))}(R)$ decreases or remains the same.
(2) If we increase f, leaving p the same, then $\lambda^{(\tau(i))}(R)$ decreases or remains the same.

Assume two tuples of position, left arc startpoint, and right arc endpoint (p, e, f) and (p', e', f') make $\lambda^{(\tau(i))}(R) = \alpha(\tau(i)$. Let the j elements be indexed j_1 through j_{i-1}. If $\tau(i)$ is in position q, then $\tau(i)$ is placed between j_q and j_{q+1}. If $p = p'$, note $e = e'$ as if $e < e'$, we have that $\lambda^{(j_e)}(R)$ changes in the two tuples and $f = f'$ as otherwise the tuples do not produce the same $\lambda^{(\tau(i))}(R)$.

Therefore, we can assume $p < p'$ without loss of generality. Note that $f > f'$. If $\lambda^{(\tau(i))}(R)$ is the same in both scenarios, then the number of j_y where $p + 1 \leq y \leq p'$ such that $j_y < \tau(i)$ is equal to the number of j_z where $f' + 1 \leq z \leq f - 1$. such that $j_z > \tau(i)$. Let y' be the first y to satisfy this condition and z' be the last z to satisfy this condition. Because $\lambda^{(y')}(R)$ cannot change between the tuples, it follows that there must be an arc betwee y' and z'. In that case f' actually should be z' as the right arc is redundant. Thus, for all $p + 1 \leq y \leq p'$ we have $j_y > \tau(i)$ and for all $f' + 1 \leq z \leq f - 1$, we have $j_z < \tau(i)$. So, when we move from the tuple (p, e, f) to the tuple (p', e', f'), we have $\lambda^{(j_y)}$ for $p + 1 \leq y \leq p'$ increases. This is a contradiction, so the tuples must be the same.

Thus, the inverse map of λ is injective as there is a unique permutation and arcs and surjective as there always is a permutation and arcs corresponding to any α. Thus, we have a bijection. We can find the number of parking functions with the following argument.

Proposition 4.6. *The number of parking functions of length n is $(n + 1)^{n-1}$.*

Proof. Consider the parking lot and parking spots. Imagine that the parking spots form a loop with points numbered from 1 to $n + 1$ counter clockwise instead of 1 to n. When we say spot l for $l > n + 1$, we are referring to the parking spot equivalent to l modulo $n + 1$. Cars enter the parking at spot 1 and move around the circle counter clockwise until they reach their preferred spot. Once there, they continue to the nearest empty spot counterclockwise. Note that the cars can always park, but their preferred spots $(a_1, \ldots, a_n)$ form a parking function if and only if the empty spot at the end of the process is numbered $n + 1$. Note that given any preferred spots $(a_1, \ldots, a_n)$ that lead to an empty spot at m, if we use preferred spots of $(a_1 + k, \ldots, a_n + k)$ for some k, note that the empty spot is not at $m + k$. Therefore, exactly one of the preferred spot tuples in $(a_1 + k, \ldots, a_n + k)$ for fixed a_i for $1 \leq i \leq n$ and variable $0 \leq k \leq n$ works. There are a total of $(n + 1)^n$ preferred spot tuples, so $1/(n + 1)$ of these tuples are parking functions. Thus, the number of parking functions is $(n + 1)^{n-1}$. ■

Thus the number of regions in the Shi arrangement is $(n + 1)^{n-1}$.

5. Finite Field Method

In the previous section, we spent a lot of effort developing a bijection to determine the number of regions in the Shi arrangement. But the bijection only told us a limited amount of information (just the number of regions but not the characteristic polynomial of the arrangement), is not generalizable to other arrangements, and also took a lot of work to prove. The finite field method fixes these problems. See [1] for more detail.

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. Let $H \in \mathcal{A}$ be a hyperplane of the form $a_1x_1 + \cdots + a_nx_n = k$. Define H_q to be

$$\{(x_1, \ldots, x_n) \in \mathbb{F}_q^n : a_1x_1 + \cdots + a_nx_n \equiv k \pmod{p}\},$$

for a prime q. Define $\mathcal{A}_q = \{H_q : H \in \mathcal{A}\}$. We can define the intersection poset $L(\mathcal{A}_q)$ as we would normally define $L(\mathcal{A})$.

Theorem 5.1. *Let $\mathcal{A}$ be an arrangement. Let q be sufficiently large prime. Then*

$$\chi_{\mathcal{A}}(q) = \left| \mathbb{F}_q^n - \bigcup_{H_q \in \mathcal{A}_q} H_q \right|.$$

Proof. We must first show that for sufficient large primes, we have $L(\mathcal{A}_q) \cong L(\mathcal{A})$. Consider some k-dimensional intersection of a subset of the hyperplanes of $\mathcal{A}$ in $\mathbb{R}^n$. We will first ensure that none of these intersections are degenerate. It follows that the intersection can be expressed as

$$\left\{ (a_1, \dots, a_n) + \sum_{i=1}^{k} x_i (v_1^{(i)}, \dots, v_n^{(i)}) : (x_1, \dots, x_k) \in \mathbb{R}^k \right\},$$

for vectors $(a_1, \dots, a_n)$ and linearly independent $(v_1^{(i)}, \dots, v_n^{(i)})$ for $1 \le i \le k$. Note that if q does not divide the determinant of the submatrix of

$$A = \begin{pmatrix} v_1^{(1)} & v_2^{(1)} & \dots & v_n^{(1)} \\ v_1^{(2)} & v_2^{(2)} & \dots & v_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{(k)} & v_2^{(k)} & \dots & v_n^{(k)} \end{pmatrix},$$

with rank equal to the rank of A, meaning the vectors $(v_1^{(i)}, \dots, v_n^{(i)})$ are linearly independent in $\mathbb{F}_q^n$, the intersection intersects q^k points in $\mathbb{F}_q^n$ because for any two distinct elements of

$$S = \left\{ (a_1, \dots, a_n) + \sum_{i=1}^{k} x_i (v_1^{(i)}, \dots, v_n^{(i)}) : (x_1, \dots, x_k) \in \mathbb{F}_q^k \right\},$$

say $(x_1, \dots, x_k) = (y_1, \dots, y_k)$ and $(x_1, \dots, x_k) = (z_1, \dots, z_k)$, we have

$$(a_1, \dots, a_n) + \sum_{i=1}^{k} y_i (v_1^{(i)}, \dots, v_n^{(i)}) - (a_1, \dots, a_n) - \sum_{i=1}^{k} z_i (v_1^{(i)}, \dots, v_n^{(i)}) = \sum_{i=1}^{k} (y_i - z_i)(v_1^{(i)}, \dots, v_n^{(i)}),$$

which cannot be 0 as the $(v_1^{(i)}, \dots, v_n^{(i)})$ are independent. There are finitely many intersections, so there are finitely many primes q which will divide the determinant of the matrix comprising of the vectors spanning the intersection. Thus, for sufficiently large q, the intersection occupies q^k points if the dimension of the intersection is k.

In order to verify $L(\mathcal{A}_q) \cong L(\mathcal{A})$, we also need to ensure that for $r, s \in L(\mathcal{A})$ and corresponding $r_q, s_q \in L(\mathcal{A}_q)$ that if $r \le s$ then $r_q \le s_q$ and if $r \not\le s$ then $r_q \not\le s_q$. Note that if $r \le s$ then $r_q \le s_q$ follows as we are just taking the intersections in $\mathbb{F}_q$. Assume that r and s are

$$(a_1, \dots, a_n) + \sum_{i=1}^{k} x_i (v_1^{(i)}, \dots, v_n^{(i)}), (b_1, \dots, b_n) + \sum_{j=1}^{l} y_j (u_1^{(j)}, \dots, u_n^{(j)}),$$

respectively. If $r \not\leq s$, then $s \not\subseteq r$, so

$$\operatorname{rank}\begin{pmatrix} v_1^{(1)} & v_1^{(2)} & \dots & v_1^{(n)} & u_1^{(1)} & u_1^{(2)} & \dots & u_1^{(n)} & b_1 - a_1 \\ v_2^{(1)} & v_2^{(2)} & \dots & v_n^{(n)} & u_2^{(1)} & u_2^{(2)} & \dots & u_2^{(n)} & b_2 - a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ v_n^{(1)} & v_n^{(2)} & \dots & v_n^{(n)} & u_n^{(1)} & u_n^{(2)} & \dots & u_n^{(n)} & b_n - a_n \end{pmatrix} > \operatorname{rank}\begin{pmatrix} v_1^{(1)} & v_1^{(2)} & \dots & v_1^{(n)} \\ v_2^{(1)} & v_2^{(2)} & \dots & v_n^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ v_n^{(1)} & v_n^{(2)} & \dots & v_n^{(n)} \end{pmatrix},$$

meaning the intersections lie in different directions or are parallel to each other. If the rank of the left matrix is t, then there must be a $t \times t$ sub-matrix with nonzero determinant. There are finitely many q which divide this determinant. Since there are finitely many pairs of intersections, for sufficiently large q, we know q will not divide any of these determinants and $L(\mathcal{A}_q) \cong L(\mathcal{A})$.

Therefore, for sufficiently large q any intersection of dimension k will occupy q^k points in $\mathbb{F}_q^n$ and the intersection poset of $\mathcal{A}_q$ will be identical to the normal intersection poset. By the formula for $\psi_{\mathcal{A}}(t) = \chi_{\mathcal{A}}(t)$, for

$$X = \mathbb{R}^n - \left(\sum_{1\le i\le k} H_i\right) + \left(\sum_{1\le i_1<i_2\le k} H_{i_1} \cap H_{i_2}\right) - \cdots = \mathbb{R}^n - \bigcup_{H\in\mathcal{A}} H,$$

if we replace every intersection with t^d where d is the dimension of the intersection, we get $\chi_{\mathcal{A}}(t)$. Using $t = q$, it follows that

$$\chi_{\mathcal{A}}(q) = \left|\mathbb{F}_q^n - \bigcup_{H_q\in\mathcal{A}_q} H_q\right|.$$

■

The way we can use Theorem 5.1 is by turning the problem of determining $\chi_{\mathcal{A}}(q)$ into a counting problem. When we solve the counting problem, we get a formula for $\chi_{\mathcal{A}}(q)$ in terms of n and q. Because the formula holds over infinitely many q, the formula must be $\chi_{\mathcal{A}}(q)$ as $\chi_{\mathcal{A}}(q)$ is a polynomial of degree n. Let us apply this method to the Shi arrangement $\mathcal{S}_n$.

Theorem 5.2. *The characteristic polynomial of $\mathcal{S}_n$ is*

$$\chi_{\mathcal{S}_n}(q) = q(q-n)^{n-1}.$$

Proof. Let q be extremely large compared to n. For the remainder of this proof, we will omit the modulo q. We have

$$\chi_{\mathcal{S}_n}(q) = |\{(x_1, \dots, x_n) \in \mathbb{F}_q^n : x_i - x_j \not\equiv 0, 1 \text{ for } 1 \le i < j \le n\}|.$$

We can think about this set as the number of ways of placing n balls numbered 1 to n on a loop with q marked points numbered 0 to $q-1$, such that all balls are on distinct marked points and two balls i and j are consecutive with j clockwise of i only if $i < j$.

Split the balls into groups where they are consecutive. Traversing the balls each group on the loop in clockwise order, they must be ascending. There must be at least one marked point between each group. Let $B_1, \dots, B_{q-n}$ set disjoint sets of the balls whose union is all of the balls. Assume without loss of generality that ball 1 is in B_1. There are q positions for ball 1 in the loop. Place the balls in B_1 in ascending order after 1. Then skip a marked point and add the balls in B_2 in ascending order. Then skip a marked point and continue. Through this process, we get a bijection between these sets with a position for ball 1 and a placement of the n balls. It follows that there are $q(q-n)^{n-1}$ ways of placing the balls, so through Theorem 5.1

$$\chi_{\mathcal{S}_n}(q) = q(q-n)^{n-1}.$$

■

With the characteristic polynomial and using Theorem 3.6, we have

Corollary 5.3. *The number of regions of $\mathcal{S}_n$ is $(n+1)^{n-1}$.*

To show the flexibility of this method, we also find the number of regions in the *Catalan arrangement*

$$\mathcal{C}_n = \{x_i - x_j = 0, \pm 1 : 1 \leq i < j \leq n\}.$$

Theorem 5.4. *The characteristic polynomial of $\mathcal{C}_n$ is*

$$\chi_{\mathcal{C}_n}(q) = q(q-n-1)(q-n-2)\cdots(q-2n+1).$$

Thus, the number of regions is $n!\,C_n$ where C_n is the nth Catalan number.

Proof. Let q be extremely large compared to n. We will omit modulo q for the remainder of the proof. We have

$$\chi_{\mathcal{C}_n}(q) = |\{(x_1, \ldots, x_n) \in \mathbb{F}_q^n : x_i - x_j \not\equiv 0, \pm 1 \text{ for } 1 \leq i < j \leq n\}|.$$

This is the number of ways to place n balls numbered 1 to n on a loop with q marked points numbered 0 to $q-1$ such that all balls are in distinct nonadjacent positions.

There are q positions for the first ball. There are $(n-1)!$ ways to order to remaining balls. Consider a sequence of q elements consisting of the n balls and $q-n$ marked points with no ball on them in the loop. Let the leftmost element of the sequence be ball 1. This corresponds to the loop when cut just before ball 1. Place n marked points after each of the balls, so that there is a marked point between each of the balls in the loop. Thus, there are a total of

$$\chi_{\mathcal{C}_n}(q) = q(n-1)!\binom{q-2n+(n-1)}{n-1} = q\frac{(q-n-1)!}{(q-2n)!} = q(q-n-1)(q-n-2)\cdots(q-2n+1).$$

ways to place n balls.

Using Theorem 3.6, we then have that the number of regions is

$$\begin{aligned}&(-1)^n \cdot -(-1-n-1)(-1-n-2)\cdots(-1-2n+1) = (n+2)(n+3)\cdots(2n),\\ =&\frac{(2n)!}{(n+1)!} = n! \cdot \frac{1}{n+1}\binom{2n}{n} = n!\,C_n.\end{aligned}$$

■

6. Coxeter Arrangements

The finite field method can also help in determining the number of regions in a Coxeter arrangement. Let us start by defining terminology. See [3] and [4] for more detail.

A *Coxeter group* is a group generated by elements $s_1, \ldots, s_n$ with identity e with the following relations

- $s_i s_i = e$.
- $(s_i s_j)^{m_{ij}} = e$ for some $2 \leq m_{ij} = m_{ji} \leq \infty$. If $m_{ij} = \infty$, then there is no relation of the form $(s_i s_j)^{m_{ij}} = e$ should be imposed. These relations are called the *braid relations*.

Example 6.1. Consider the Coxeter group with generators s and t and identity e with the braid relation $(st)^3 = e$. This braid relation can be rewritten as $sts = tst$. See Figure 7 for a depiction of the group.

e

s t

st ss tt ts

sss sts tst ttt

stt tts sst tss

$\cdots$ $stss$ $tsts$ $stst$ $ttts$ $\cdots$

Figure 7. Words of the Coxeter group with generators s and t and identity e with the braid relation $(st)^3 = e$. The colors depict the equivalence classes in the words.

This group can be constructed in a different way. Consider a triangle as depicted in Figure 8 with reflection moves s and t. We have $ss = e$, and we also know that st is a rotation by 120° clockwise, so $(st)^3 = e$. This technique of looking at Coxeter groups as the reflections of a shape applies more generally with the use of higher-dimensional shapes. The condition of $s_i s_i = e$ means

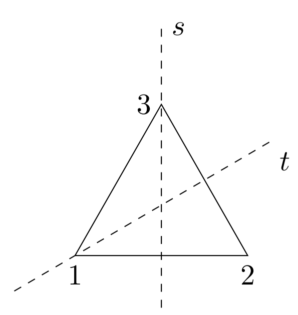


Figure 8. A triangle and two reflective axes of symmetry: s and t.

that s_i is a reflection and the condition of $(s_is_j)^{m_{ij}} = e$ can be interpreted as s_is_j is a rotation and it takes m_{ij} rotations to bring the shape back to its normal orientation, so the angle between the hyperplanes which induce reflection of s_i and s_j is $2\pi/m_{ij}$. Removing the shape and just keeping the normal vectors of the reflecting planes, we get a *root system* Δ consisting of a finite set of vectors in $\mathbb{R}^n$ for some n such that

- For all $\alpha \in \Delta$, we have $\Delta \cap \{x\alpha : x \in \mathbb{R}\} = \{-\alpha, \alpha\}$.
- Reflecting over the linear hyperplane normal to α maps Δ to itself.

The main question we will ask is how many regions do the hyperplanes normal to the vectors of the roots system form. This question has a general solution using the finite field method (see [1] for more details). We will examine two infinite classes of Coxeter groups and their corresponding arrangements. Using the finite field method, we can find the number of regions.

6.1. **Coxeter Groups of Type A.** In Figure 8, we saw that the symmetries of a triangle formed a Coxeter group. By labeling the vertices as shown on the figure, the group is actually the permutation group on three numbers. For example, s corresponds to the permutation 213 and t corresponds to the permutation 132, i.e., swapping the first two elements and second two elements. These permutations extend naturally to 4 elements where the permutations $s = 2134$, $t = 1324$, and $u = 1243$ are the generators. Here the braid relations are $(st)^3 = e$, $(tu)^3 = e$, and $(su)^2 = e$. In general, for n length permutations the generators $s_1, \ldots, s_{n-1}$ follow the braid relations $(s_is_j)^3 = e$ if i and j are consecutive and $(s_is_j)^2 = e$ otherwise. This Coxeter group is referred to as A_{n-1}.

The group of permutations of length n can be expressed as the symmetries of an n-simplex, i.e., the polytope with vertices at the unit coordinate vectors $\mathbf{e}_i$ for $1 \leq i \leq n$ in $\mathbb{R}^n$. The generator permutations correspond to reflections over the hyperplanes $x_i - x_{i+1} = 0$ for $i = 1, \ldots, n-1$ as these reflections swap the ith coordinate with the $(i+1)$st coordinate and leave all of the other coordinates unchanged. The root system generated by the normal vectors to these hyperplanes is $\mathbf{e}_i - \mathbf{e}_j = 0$ for $1 \leq i < j \leq n$, so the hyperplanes formed by the root system is the Braid arrangement $\mathcal{B}_n$. In Section 4, we found that the number of regions in the arrangement was $n!$.

6.2. **Coxeter Groups of Type B.** Consider the following root system Δ with vectors $\pm e_i$ for $1 \leq i \leq n$ and $\pm e_i \pm e_j$ for $1 \leq i < j \leq n$. To show this is a root system, if we reflect it over $x_i = 0$, the vectors of the form $\pm e_j = 0$ and $\pm e_j \pm e_k$ for $j, k \neq i$ are mapped to themselves, the vector e_i and $-e_i$ are interchanged, and the vectors of the form $\pm e_i \pm e_j$ are mapped to $\mp e_i \pm e_j$. for $1 \leq i \neq j \leq n$.

If we reflect over the hyperplane $x_i - x_j = 0$ on the other hand, we are swapping the ith and jth coordinate. Thus, $\pm e_k$ and $\pm e_k \pm e_l$ for $k, l \neq i, j$ are mapped to themselves while $\pm e_i$ and $\pm e_j$ are interchanged and $\pm e_i \pm e_k$ and $\pm e_j \pm e_k$ are interchanged for $k \neq i, j$.

If we reflect over the hyperplane $x_i + x_j = 0$, we are swapping and inverting the ith and jth coordinate. Thus, $\pm e_k$ and $\pm e_k \pm e_l$ for $k, l \neq i, j$ are mapped to themselves while $\pm e_i$ and $\mp e_j$ are interchanged and $\pm e_i \pm e_k$ and $\mp e_j \pm e_k$ are interchanged for $k \neq i, j$.

Therefore Δ is indeed a root system, and we refer to the Coxeter group represented by the system as B_n. This family of Coxeter groups turn out to be the symmetries of a hypercube, i.e., polytopes with vertices at $S = \{\sum_{i=1}^{n} \pm \mathbf{e}_i\}$.

With these hyperplanes, we can now determine the number of regions in B_n. We have

$$\chi_{B_n}(q) = \{(x_1, \ldots, x_n) \in \mathbb{F}_q^n : x_i \not\equiv 0, \pm x_j \text{ for } 1 \leq i, j \leq n\}.$$

There are $q-1$ choices for x_1 as x_1 cannot be 0. There are then $q-3$ choices for x_2 as x_3 cannot be $0, \pm x_2$. Continuing, we see that the number of ways to pick $(x_1, \ldots, x_n)$ is

$$\chi_{B_n}(q) = (q-1)(q-3)\cdots(q-(2n-1)),$$

so the number of regions is

$$(-1)^n(-1-1)(-1-3)\cdots(-1-(2n-1)) = (2)(4)\cdots(2n) = 2^n n!\,.$$

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