Linear Programming and Its Applications

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Why I Chose This Topic

- Many math research papers are highly niche and lack real-world connection.
- These topics often feel distant unless pursued academically long-term.
- I wanted something broadly relatable—tied to real-life experiences.
- But I also wanted mathematical depth and complexity.
- Linear programming offers both: real applications and rich theory.
- Today's talk is high-level, but my paper includes deeper proofs, algorithms, and extended applications.

Introduction

What is Linear Programming (LP)?

- Linear Programming (LP) optimizes a linear objective function subject to linear constraints.

Example: Shipping Cost Minimization

A company ships from 2 factories (F1, F2) to 3 warehouses (W1, W2, W3).

Cost matrix:

	W1	W2	W3
F1	4	6	8
F2	5	4	3

F1 can ship 70 units, F2 can ship 30. Warehouse demands: W1 (30), W2 (30), W3 (40).

Solving the Shipping Problem

Objective: Minimize total cost.
Let x_{ij} be units shipped from factory i to warehouse j.
Formulation:

$$\min \; 4x_{11} + 6x_{12} + 8x_{13} + 5x_{21} + 4x_{22} + 3x_{23}$$

Subject to:

$$\begin{aligned} x_{11} + x_{12} + x_{13} &= 70 \\ x_{21} + x_{22} + x_{23} &= 30 \\ x_{11} + x_{21} &= 30 \\ x_{12} + x_{22} &= 30 \\ x_{13} + x_{23} &= 40 \\ x_{ij} &\geq 0 \end{aligned}$$

Optimal Solution: $x_{11} = 30, \; x_{12} = 30, \; x_{23} = 40$ with cost = \$420.

Example: Diet Optimization

Goal: Choose two foods to satisfy nutrition at minimal cost.
Variables: x_1 = servings of Food A, x_2 = servings of Food B
Constraints:

$$\begin{aligned} 2x_1 + x_2 &\geq 30 \quad \text{(Protein)} \\ 4x_1 + 3x_2 &\geq 50 \quad \text{(Carbs)} \\ x_1 + 2x_2 &\leq 40 \quad \text{(Fat)} \\ x_1, x_2 &\geq 0 \end{aligned}$$

Objective: Minimize $1.5x_1 + 2x_2$

Diet Problem Solution (Graphical)

- Feasible region formed by intersecting constraints.
- Corner points evaluated:

$$\text{Optimal at } x_1 = 10, x_2 = 10 \Rightarrow \text{Cost} = 1.5(10) + 2(10) = 35$$

Interpretation: Eat 10 servings of each food to minimize cost.

Historical Background

- **1939:** Kantorovich optimizes Soviet industrial planning with LP.
- **1947:** George Dantzig invents the Simplex method for military logistics.
- LP grows into a cornerstone of modern optimization.

Today: LP powers algorithms in scheduling, AI, economics, and robotics.

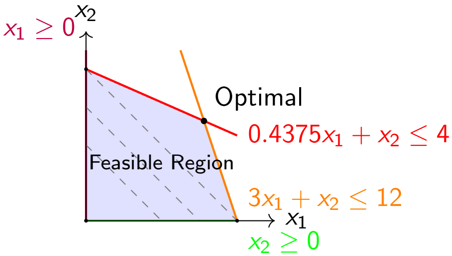
Real-World Motivations

- **Logistics:** Minimize transportation costs from factories to warehouses.
- **Finance:** Allocate investments to maximize returns under risk.
- **Robotics:** Plan robot arm movements while avoiding obstacles.
- **Manufacturing:** Maximize profit given resource and labor limits.

Mathematical Foundations

Geometric Interpretation

- LP feasible region is a convex polyhedron formed by intersecting half-spaces.
- The optimal solution lies at a vertex of this region.
- Convexity ensures any local optimum is global.
- Constraints define half-planes $\rightarrow$ intersection is convex region.
- LP objective is linear $\rightarrow$ best value lies on a vertex.
- You don't need to test every point, just the corners.



The Simplex Method

- Starts at an initial vertex (basic feasible solution).
- Moves along edges improving the objective.
- Stops when no direction improves $\rightarrow$ optimal vertex.
- Finite, but worst-case exponential (Klee-Minty).

Theorem: *If the simplex algorithm terminates at a basic feasible solution, that solution is optimal.*

Klee-Minty Cube: Worst Case for Simplex

- LP where Simplex visits all 2^n vertices.
- In 3D:

$$\begin{aligned} 0 \le x_1 \le 1 \\ \epsilon x_1 \le x_2 \le 1 - \epsilon x_1 \\ \epsilon x_2 \le x_3 \le 1 - \epsilon x_2 \end{aligned}$$

- **Objective:** Maximize x_3

Lesson: LP is easy in practice, but simplex isn't always fast in theory.

Klee-Minty Cube: Diagram

- Constructs distorted hypercube with exponential path for simplex.
- Demonstrates exponential worst-case.

Theorem: *There exist LPs where simplex visits all 2^n vertices.*

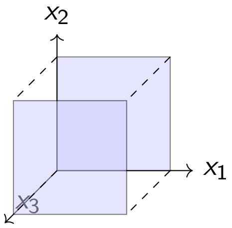


Figure 1: Distorted hypercube for the Klee-Minty LP example in three dimensions.

Interior-Point Methods

- Introduced by Karmarkar (1984), polynomial time.
- Stays within the interior using log barriers.
- Solves:

$$\min \; c^T x - \mu \sum \log(x_i)$$

- Great for large, sparse problems.

Duality: Two Perspectives

- **Primal:** Maximize $c^T x$ subject to $Ax \leq b$
- **Dual:** Minimize $b^T y$ subject to $A^T y \geq c$
- **Strong duality:** Optimal values match: $c^T x^* = b^T y^*$

Duality: Visualization

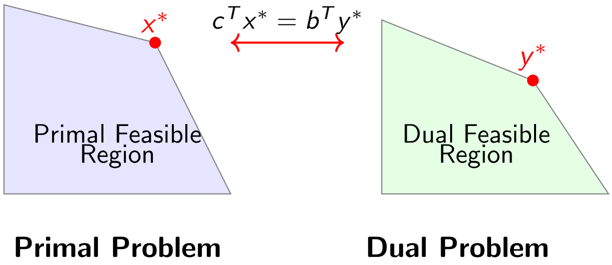


Figure 2: Conceptual illustration of primal-dual relationship. Strong duality ensures that optimal values coincide.

Modern Technology

Modern Applications

- **Logistics:** Delivery routing, freight assignment.
- **Robotics:** Motion planning, force control.
- **Machine Learning:** LPBoost, fairness constraints.
- **Networks:** Flow optimization, resource scheduling.

Solver Technologies

- **Gurobi, CPLEX:** Commercial state-of-the-art solvers.
- **HiGHS, GLPK:** Open-source alternatives.
- **PDLP:** Google's distributed LP solver using GPUs.
- Techniques: Presolve, parallelism, warm-starts.

Open Problems

Open Problems and Frontiers

- Is there a strongly polynomial Simplex algorithm?
- How far can quantum solvers take us?
- Can we bridge LP and integer programming more tightly?

Conclusion

Conclusion

- Linear programming blends practical utility with elegant math.
- It powers systems from logistics to AI.
- And it continues to raise questions that drive new research.

Thank you! Questions?