Introduction to Group cohomology

Marigold Strupp

July 14, 2025

What is group cohomology?

- group cohomology is a mathematical tool that gives one a cohesive framework for understanding how a group G acts on an abelian group A.
- It does this via associating to each such action to whats called a cochain complex, essentially a sequence of abelian groups and homomorphisms.
- We then take something called the cohomology of this cochain complex to measure it's local properties.

G-modules

The basic object of study in group cohomology are G-modules. Letting G be any group, a G-module is simply an abelian group A equipped with a G action such that for any $g \in G$ and $a, b \in A$ we have

$$g(a + b) = ga + gb$$

A G-module homorphism $f : A \to B$ is simply an abelian group homomorphism with the additional condition that

$$f(ga) = gf(a)$$

Example

- Let $\phi : G \to GL_n(\mathbb{R})$ be a representation of G.
- Then for any $g \in G$ and any vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ $\phi(g)(\vec{v} + \vec{w}) = \phi(g)\vec{v} + \phi(g)\vec{w}$.
- Meaning a representation is just a G-module in disguise!

G-modules

Example

- Let S be a G-set.
- Let $\mathbb{Z}[S]$ be the free abelian group generated by S. That is, the abelian group consisting of formal sums of $s \in S$.
- Turn this into a G-module via defining for each $g \in G$

$$g \sum_{s \in S} n_s s := \sum_{s \in S} n_s gs$$

- If S is a G set and A is a G-module then we have a one to one correspondence between G-set homomorphisms $S \to A$ and G-module homomorphisms $\mathbb{Z}[S] \to A$.

Cochain complex

- To any G-module A we associate for each $i \in \mathbb{Z}$ G-modules $C^i(G,A)$ and homomorphisms $\partial^i : C^i(G,A) \to C^{i+1}(G,A)$.
- For negative $i \in \mathbb{Z}$ we define $C^i(G,A) := 0$, otherwise $C^i(G,A) := \{f : G^i \to A\}$.
- We define $\partial^i : C^i(G,A) \to C^{i+1}$ by:

$$(\partial^i f)(g_1,\dots,g_{i+1}) := g_1 f(g_2,\dots,g_{i+1}) + \sum_{n=1}^{i}(-1)^n f(g_1,\dots,g_k g_{k+1},\dots,g_i) + (-1)^i f(g_1,\dots,g_{i+1})$$

- A simple, albeit long, calculation shows that $\mathrm{Img}(\partial^{i-1}) \subset \mathrm{Ker}(\partial^i)$

Cohomology

- We define $Z^i(G, A) := \mathrm{Ker}(\partial^i)$ and $B^i(G, A) := \mathrm{Img}(\partial^{i-1})$.
- The ith Cohomology group is then defined as $H^i(G, A) := Z^i(G, A)/B^i(G, A)$

Example

- Note that $C^0(G, A)$ consists of functions from G^0 (ie. a set with one element) to A and thus is just A.
- Now realize that $C^{-1}(G, A) = 0$ and thus that $B^0 = 0$.
- Then $H^0(G, A) = Z^0(G, A)$, which using the definition from the previous slide we can calculate to consist of those elements of $a \in A$ such that for all $g \in G$:

$$ga - a = 0$$

- Thus $H^0(G, A)$ is the abelian group of "invariants" of A, that is, the maximal subgroup of A for which G acts trivially.

An explicit description of H^2

- Let G be any group and A any G-module
- Then we have

$$(\partial^2 f)(g_1, g_2, g_3) = g_1 f(g_2, g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f(g_2, g_3)$$

- So $Z^2(G, A)$ consists of those functions $f : G^2 \to A$ such that

$$g_1 f(g_2, g_3) = f(g_2, g_3) + f(g_1 g_2, g_3) - f(g_1, g_2 g_3) \quad (0.1)$$

- We also have that

$$(\partial^1 f)(g_1, g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_2)$$

- So $B^2(G, A)$ consists of functions $f : G^2 \to A$ such that there is some $h : G \to A$ with:

$$f(g_1, g_2) = g_1 h(g_2) - h(g_1 g_2) + h(g_2) \quad (0.2)$$

- So $H^2(G, A)$ consists of functions $f : G^2 \to A$ satisfying (0.1) modulo functions of the form described in (0.2).

Extensions

We will now discuss a fairly simple application of the second cohomology group to something called the extension problem. The extension problem essentially asks how many ways there are to combine two groups. To be more precise, an extension of G by A is an exact sequence:

$$0 \longrightarrow A \xrightarrow{j} E \xrightarrow{\pi} G \longrightarrow 0$$

In other words, E has A as a normal subgroup and G as the quotient E/A.

Example

If we have a group A and a group G, consider their product $A \times G$ is an extension. The injection $i : A \to A \times G$ is given by $i(a) := (a, 1)$. The surjection $\pi : A \times G \to G$ is given by $\pi(a, g) := g$.

Equivalent Extensions

Importantly we consider two extensions

$$0 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 0$$

and

$$0 \longrightarrow A \longrightarrow F \longrightarrow G \longrightarrow 0$$

equivalent if we have a commutative diagram:

$$\begin{array}{ccccccc} & & & E & & & \\ & & \nearrow & \downarrow & \searrow & & \\ 0 \longrightarrow & A & & \varphi & & G & \longrightarrow 0 \\ & & \searrow & \downarrow & \nearrow & & \\ & & & F & & & \end{array}$$

Where φ is an isomorphism. The extension problem is to classify extensions up to equivalence.

An action from an extension

- Let G be a group and A an abelian group. Let

$$0 \longrightarrow A \longrightarrow X \xrightarrow{\pi} G \longrightarrow 0$$

be an extension.

- If we have an element $g \in G$ and an element $a \in A$ then we have a well defined action by g on a given by $xj(a)x^{-1}$ where $x \in X$ is such that $\pi(x) = g$.
- We will denote the action of an element $g \in G$ on an element $a \in A$ by $\theta_g(a)$. We denote the G module with A as it's underlying abelian group A_θ

Cohomology class from an extension

- Let
$$0 \longrightarrow A \longrightarrow X \xrightarrow{\pi} G \longrightarrow 0$$
be an extension
- Now let $r : G \to X$ be a set theoretic section (that is, a set function such that $\pi \circ r = id$.)
- Define $R : G^2 \to A$ by $R(g,h) = r(g)r(h)r(gh)^{-1}$.
- This is easily checked to be a cocycle meaning $R \in Z^2(G, A_\theta)$.
- Let s be another set function such that $\pi \circ s = id$, and S is it's corresponding cocycle. Then a fairly simple calculation shows that $R - S \in R \in B^2(G, A_\theta)$.
- We thus have a well defined assignment F, sending an extension to $[R]$ where $[R]$ is the cohomology class of the cocycle induced by some section r of π.

Extension from a Cohomology class

- Let G be a group and A a G-module, let $\theta_g(a)$ denote the action of an element $g \in G$ on $a \in A$. Suppose $R \in Z^2(G, A)$. We will give an assignment H from $H^2(G, A)$ to extensions of G by A.
- Define X_R to be a group with underlying set the cartesian product $A \times G$ and multiplication defined by $(a, g) * (b, g) = (\theta_g(b) + a + R(g, h), gh)$.
- The identity is then the element $(-R(1, 1), 1)$. Inverses are therefore given by $(a, b)^{-1} = (-\theta_{g^{-1}}(R(g, g^{-1}) + R(1, 1) + a), g)$
- as for the extension. The injection $i : A \to X_R$ is given by $i(a) := (a - R(1, 1), 1)$ and the surjection $\pi : X_R \to G$ is given by $\pi(a, g) := g$.
- Finally if $R - S = D \in B^2(G, A)$ then they give equivalent extensions via the bijection $\varphi : X_R \to X_S$ defined by $f(a, g) := (a + D(g), g)$.

These two assignments are inverse to one another

- Let X be the extension:

$$0 \longrightarrow A \longrightarrow X \xrightarrow{\pi} G \longrightarrow 0$$

  be an extension such that A is abelian. Denote the induced action of $g \in G$ onto $a \in A$, $\theta_g(a)$, Let R denote the cocycle induced by a section r of π.

- We now produce an isomorphism $\varphi : X \to X_R$ by $\varphi(x) := (xr(\pi(x))^{-1}, \pi(x))$. This is an isomorphism because it has an inverse given by $\varphi^{-1}(a, g) := ar(g)$,

- It turns out, that this is not just isomorphism but an equivalence of extensions.

- Thus the two assignments give identical extensions up to equivalence.

Further applications

- We have just shown one particuar application, namely that H^2 classifies extensions, but there are many more.
- In number theory one can characterize the Brauer groups with cohomology.
- In topology group cohomology is actually equivalent to the cohomology of a certain type of topological space called classifying spaces.
- Another application to group theory is to the study of cyclic groups and their G-modules. In particular, the cohomology of cyclic groups is periodic.