A SHORT MARCH THROUGH GROUP COHOMOLOGY

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Abstract. This paper provides a concise introduction to cohomology, a tool that comes up constantly in mathematics. We define homology and cohomology, discuss its low dimensional cases, how one uses the second cohomology group to classify extensions of abelian kernel. We finish with an introduction to a few more advanced structural tools that apply to cohomology.

1. Introduction

Group cohomology is a rich and flexible invariant that arises when one studies group actions through the lens of homological algebra. It appears throughout mathematics, with applications ranging from number theory to topology. At its core, group cohomology provides a systematic framework for understanding how a group G acts on an abelian group A. It assigns to each such pair two sequences of abelian groups $H^n(G, A)$ and $H_n(G, A)$ which encode deep information about the relationship between G and A. While these groups can be defined algebraically via derived functors, they admit remarkably concrete interpretations in low dimensions.

This paper offers a concise yet meaningful introduction to group cohomology. We introduce the basic definitions and theorems of the theory while giving examples and a few applications along the way. One such application discussed in detail is the classification of group extensions with abelian kernel.

We begin with the necessary background. The reader is assumed to be familiar with elementary group, ring, module, and category theory, but not with homological algebra. The first section therefore introduces tensor products, chain complexes, and resolutions, as well as G-modules and the group ring.

The next section defines group homology and cohomology as the derived functors of coinvariants and invariants, respectively. To make these constructions concrete, we introduce the bar resolution, a hands-on method for computing both homology and cohomology that reveals how the formal machinery translates into explicit algebraic data.

After that we examine low-dimensional cohomology in detail. This is where the theory becomes particularly intuitive and applicable. For instance, the first cohomology group H^1 describes certain automorphisms of the affine group over a commutative ring, while the second cohomology group H^2 classifies group extensions-capturing the ways in which two groups can be combined.

The final section surveys the rich structural properties of group cohomology. We discuss, for example, long exact sequences of homology and cohomology as well as the specific case of tate cohomology. Emphasis is placed on the most commonly used tools in practice. Although many additional tools exist, we limit our focus to those that arise most frequently in applications.

2. Algebraic preliminaries

2.1. **Some homological algebra.** Despite this ostensibly being a paper about group cohomology, we will not begin by talking about groups, we actually need to start by talking a bit about modules. Most of what I'll be talking about in this subsection falls in the broader landscape of homological algebra, a subject that deals in much more generality than is necessary for the subject matter of this paper. In particular I am not covering projective objects, injective objects, and right resolutions. Throughout, let R be a ring and assume all rings are unital.

We first have to define the tensor product. This is just a way to make a space $A \otimes_R B$ such that R-linear functions out of $A \otimes_R B$ correspond to bilinear functions out of $A \times B$

Definition 2.1. For any (left) R-module A and (right) R-module B. The tensor product $A \otimes_R B$ is the abelian group generated by the symbols $a \otimes b$ where $a \in A$ and $b \in B$ modulo the following relations:

- $a \otimes b + c \otimes b = (a + c) \otimes b$
- $a \otimes b + a \otimes d = a \otimes (b + d)$
- $ra \otimes b = a \otimes br$

Notably the tensor product is functorial: If $A \xrightarrow{f} B$ is a homomorphism of modules then for any module C we have an induced homomorphism $A \otimes_R C \xrightarrow{f \otimes_R C} B \otimes_R C$ defined by

$$f \otimes_R C(a \otimes c) = f(a) \otimes c.$$

The tensor product is also commutative. Meaning $A \otimes_R B \cong B \otimes_R A$. This just comes from the isomorphism $f(a \otimes b) := b \otimes a$. Additionally it is particularly nice with free modules.

Lemma 2.2. *For any free module F on some set S and any module A, $A \otimes_R F \cong \bigoplus_S A$. Or, to be slightly more precise. For any $s \in S$ let $[s]$ be the corresponding generator in F. Then we have that any element of $x \in F \otimes_R A$ is of the form $\sum_{s \in S} m_s([s] \otimes a)$.*

Proof. By definition every element of $x \in F \otimes_R A$ may be written as

$$\begin{aligned}
\sum_{f \in F} c_f(f \otimes a) &= \sum_{f \in F} c_f((\sum_{s \in S} b_s[s])) \otimes a) \\
&= \sum_{f \in F} \sum_{s \in S} c_f b_s([s] \otimes a) \\
&= \sum_{s \in S} m_s([s] \otimes a)
\end{aligned}$$

Thus $F \otimes_R A \cong A^n$ with every element of the form $\sum_{s \in S} m_s([s] \otimes a)$. ■

Definition 2.3. A chain complex is a sequence of modules and homomorphisms:

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots$$

such that $\partial_n \circ \partial_{n+1} = 0$. That is, $\mathrm{Img}(\partial_{n+1}) \subset \ker(\partial_n)$. We call the kernels of these homomorphisms cycles and denote the cycles by $Z_n(C_*) := \ker \partial_n$. We call the images boundaries and denote boundaries by $B_n := \mathrm{Img}(\partial_{n+1})$ In particular a chain complex is said to be exact if $\mathrm{Img}(\partial_{n+1}) = \ker(\partial_n)$. A cochain complex is identically defined except everything is indexed in the opposite direction. So a cochain complex looks like:

$$\dots \longrightarrow C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \longrightarrow \dots$$

. We call the kernels cocycles and denote them $Z^n(C^*) := \ker(\partial^n)$. We call the images coboundaries and denote $B^n(C^*) := \mathrm{Img}(\partial^{n-1})$.

In order to measure the failure of a chain complex to be exact we use homology.

Definition 2.4. We define the nth homology group of a chain complex by $H_n(C_*) := Z_n(C_*)/B_n(C_*)$. The nth cohomology group of a cochain complex is defined identically by $H^n(C^*) := Z^n(C^*)/B^n(C^*)$.

Exactness is an incredibly strong property. It can be used to characterize all sorts of things. For example every homomorphism of modules decomposes into an exact sequence:

$$0 \longrightarrow \ker(f) \longrightarrow A \xrightarrow{f} B \longrightarrow B/\mathrm{Img}(f) \longrightarrow 0$$

Additionally we can obtain a long exact sequence from two shorter exact sequences.

Lemma 2.5 (Snake lemma)**.** *Suppose the following diagram commutes and has exact rows:*

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

Then we have a long exact sequence

$$\ker(a) \longrightarrow \ker(b) \longrightarrow \ker(c) \longrightarrow \mathrm{Coker}(a) \longrightarrow \mathrm{Coker}(b) \longrightarrow \mathrm{Coker}(c)$$

The proof of the snake lemma is long and cumbersome but ultimately fairly trivial. I urge the reader to attempt a proof on their own however a detailed proof may be found in [5].

Presentations are also just exact sequences in disguise. Every presentation of a given abelian group A is just an exact sequence:

$$0 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow A \longrightarrow 0$$

where F_1 and F_2 are free. In fact this generalizes to all modules via the idea of a free resolution.

Definition 2.6. A free resolution of a given module A is an exact chain complex:

$$\ldots \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow A \longrightarrow 0$$

such that each F_n is free.

Free resolutions allow us to give a nice description of the structure of a module. F_1 gives generators, F_2 relations among generators, F_3 relations among relations and so on.

We are also able to describe what are called exactness properties.

Definition 2.7. A covariant additive functor F is said is said to be:

- left exact if and only if exactness of $0 \to A \to B \to C$ implies $0 \to FA \to FB \to FC$ is exact.
- right exact if and only if exactness of $A \to B \to C \to 0$ implies $FA \to FB \to FC \to 0$ is exact.

And a contravariant additive functor F is said to be:

- left exact if and only if exactness of $A \to B \to C \to 0$ implies $0 \to FC \to FB \to FA$ is exact.

- right exact if and only if exactness of $0 \to A \to B \to C$ implies $FC \to FB \to FA \to 0$ is exact.

Example. Both Hom and $\otimes_R$ fulfill certain exactness properties.

- For any R-module M, $\mathrm{Hom}(-, M)$ is a contravariant left exact functor and $\mathrm{Hom}(M, -)$ is a covariant left exact functor. In particular if M is free then $\mathrm{Hom}(M, -)$ is both left and right exact.
- For any R-module M, $- \otimes_R M$ is right exact. Similarly to Hom if M is free $- \otimes_R M$ is both right and left exact.

The proofs of these facts are not particularly difficult but not terribly illuminating, one can find such proofs in [4].

We are finally able to define derived functors. The essential idea is that we extend a functor to a sequence of functors via a resolution.

Definition 2.8. Let F be a right exact covariant functor, or left exact contravariant functor. Then for any module A let C_n be a free resolution of A. Then the nth derived functor DF^n is defined as the nth homomology group of the chain complex FC_n.

Via some homological algebra one can show that derived functors are independent of choice of free resolution and that $DF^0 = F$. [4]

Example. The two canonical examples of derived functors are Ext and Tor. $\mathrm{Ext}^n(-, B)$ is defined to be the nth derived functor of $\mathrm{Hom}(-, B)$ and $\mathrm{Tor}^n(-, B)$ is defined to be the nth derived functor of $- \otimes_R B$. We can also define derived functors of $\mathrm{Hom}(B, -)$ using something called injective resolutions. It turns out that these agree with Ext, meaning $D\,\mathrm{Hom}(B, -)^n(A) = \mathrm{Ext}^n(B, A)$. [4]

These wind up being the only derived functors we need in this paper.

2.2. **G-Modules and The group ring.** The central objects studied by group cohomology are of course groups, as well as what are called G-modules.

Definition 2.9 (G-Module)**.** Let G be a group. Then a G-module A is simply an abelian group equipped with a G action such that for all $g \in G$ and $a, b \in A$ $g(a + b) = ga + gb$. A G-module homomorphism $\phi : A \to B$ is simply a group homomorphism such that for all $a \in A$, $\phi(ga) = g\phi(a)$. We denote the category of G-modules G-Mod.

One very nice example of G-modules are free G-modules on G-sets.

Example. For any G-set S we have a corresponding G-module $G[S]$ with

- underlying abelian group equal to the free abelian group on S. That is, every element is a formal sum of elements of $s \in S$;
- scalar multiplication is given, for any $g \in G$ by:

$$g \sum_{s \in S} n_s s := \sum_{s \in S} n_s g s$$

G-Mod is the primary category where group cohomology takes place. In particular, we always take the cohomology of a given group G with respect to a choice of G module. Group cohomology is really all about the ways in which a group G and a given G-module are related. The use of the word module is immediately justified by the following definition and lemma.

Definition 2.10 (The group ring)**.** Let G be a group and define $\mathbb{Z}[G]$ to have underlying abelian group:

$$\bigoplus_{g \in G} \mathbb{Z}$$

Meaning elements of $\mathbb{Z}[G]$ look like formal sums of elements of G. With multiplication defined by

$$\left(\sum_{g \in G} a_g g\right)\left(\sum_{g \in G} b_g g\right) = \sum_{g \in G}\sum_{h \in G} a_h b_{h^{-1}g} g$$

Notably it immediately follows from the above definition that for any $g, h \in G$, their multiplication in $\mathbb{Z}[G]$ is equal to their multiplication in G.

Lemma 2.11. *The categories G-mod and $\mathbb{Z}[G]$-mod are equivalent.*

Proof. Note first that for any $\mathbb{Z}[G]$ module A, any $g, h \in G$, and any $a, b \in A$ we have:

1. $g(ha) = (gh)a$
2. $g(a + b) = ga + gb$
3. $gh \in G$

Thus any $\mathbb{Z}[G]$-module gives a group action on A respecting its abelian group structure. In other words any $\mathbb{Z}[G]$-module gives a G-module.

Now note that any element of $\mathbb{Z}[G]$ is of the form $\sum_{g \in G} a_g g$ and thus the structure of any G-module A is determined entirely by the underlying abelian group of A and the action of G on A. So any G-Module gives a $\mathbb{Z}[G]$ module.

Now that we have a one to one correspondence of objects we just have to show that this correspondence respects morphisms. If $\varphi : A \to B$ is a $\mathbb{Z}[G]$ module homomorphism then $\varphi(ga) = g\varphi(a)$ and is a homomorphism of G-modules. Now if $\varphi : A \to B$ is a G-module homomorphism it is also an abelian group homomorphism and thus

$$\varphi\left(\sum_{g \in G} a_g(ga)\right) = \sum_{g \in G} a_g \varphi(ga) = \sum_{g \in G} a_g g \varphi(a)$$

Therefore φ is a $\mathbb{Z}[G]$-module homomorphism. Thus $\mathbb{Z}[G]$-mod and G-mod are equivalent. ∎

This lemma provides us with the ability to take tensor products, homomorphisms, and free resolutions. In particular two constructions are of high interest to us.

Definition 2.12 (Invariants and Coinvariants)**.** Every G-module A has two very important abelian groups associated to it:

- The invariants of A, denoted A^G, the subgroup of all elements $a \in A$ such that for all $g \in G$ $ga = a$.
- The coinvariants, denoted A_G, the quotient A/X where X is the subgroup of A generated by elements of the form $ga - a$ for some $a \in A$.

These two correspond to a certain homomorphism group and tensor product respectively.

Lemma 2.13. *Let A be a G-module and $\mathbb{Z}_{triv}$ denote the integers with the trivial group action $gn = n$ for all $g \in G$, then:*

- $A_G \cong \mathbb{Z}_{triv} \otimes_{\mathbb{Z}[G]} A$

- $A^G \cong \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}_{triv}, A)$

Proof.
- Let $x \in \mathbb{Z}_{triv} \otimes_{\mathbb{Z}[G]} A$ then

$$x = \sum_{k=1}^{n} m_k \otimes a_k = \sum_{k=1}^{n} m_k(1 \otimes a_k) = \sum_{k=1}^{n} 1 \otimes m_k a_k = 1 \otimes \sum_{k=1}^{n} m_k a_k$$

  Thus all elements of $\mathbb{Z}_{triv} \otimes_{\mathbb{Z}[G]} A$ are of the form $1 \otimes a$ where $a \in A$. We define a map $\varphi : \mathbb{Z}_{triv} \otimes_{\mathbb{Z}[G]} A \to A_G$ by $\varphi(1 \otimes a) := [a]_{A_G}$. Then if $1 \otimes a = 1 \otimes b$ we must have $a = gb$ where g is acts trivial on $\mathbb{Z}_{triv}$. By definition of $\mathbb{Z}_{triv}$, g could be any element of G. Thus $b = ga$ meaning $[b]_{A_G} = [a]_{A_G}$. Therefore φ is well defined. It is clearly surjective because if $[a]_{A_G} \in A_G$, then $1 \otimes a \in \mathbb{Z}_{triv} \otimes A$ and $\varphi(1 \otimes a) = [a]_{A_G}$. Furthermore if $1 \otimes a \neq 1 \otimes b$ then $b \neq ga$ meaning

$$\varphi(1 \otimes a) = [a]_{A_G} \neq [b]_{A_G} = \varphi(1 \otimes b)$$

  meaning φ is injective. Since φ is both surjective and injective, it is bijective and thus an isomorphism.
- Let $x \in \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}_{triv}, A)$ then for all $n \in \mathbb{Z}$ $x(n) = nx(1)$ and is thus entirely determined by where it sends 1. Therefore we may define an injective module homomorphism $\phi : \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}_{triv}, A) \to A$ by $\phi(x) = x(1)$. The only other condition on x is that for all $g \in G$ $x(g1) = gx(1)$ but since $g1 = 1$, this condition is equivalent to requiring $x(1) = gx(1)$ or in other words requiring $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}_{triv}, A) \cong \text{Img}(\phi) = A^G$. ∎

3. Group (co)homology

3.1. **Definitions.**

Lemma 2.13 is wonderful not only because it nicely characterizes A_G and A^G but also because it tells us that they are right and left exact functors respectively. This means we can finally define group homology and cohomology.

Definition 3.1. Let G be a group and A a G-module.

- The nth homology functor of G, denoted $H_n(G, -)$ is the nth derived functor of the coinvariant functor. In light of lemma 2.13 we can interpret this as saying that the nth homology group of G with respect to A, $H_n(G, A)$ may be defined as $\text{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}_{triv}, A)$.
- The nth cohomology functor of G, denoted $H^n(G, -)$ is defined to be the nth derived functor of the invariant functor. Similarly to homology, this can be interpreted using lemma 2.13 say that the nth cohomology group of G with respect to A, $H^n(G, A)$ is defined to be $\text{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}_{triv}, A)$.

These are the formal definitions of group homology and cohomology. However, taken on their own, they offer little intuition. If a random person on the street asked you what group homology and cohomology actually "mean", you'd likely struggle to provide a satisfying answer. The rest of this section will be dedicated to equipping you with the intuition and information that might allow you to, at least partially, answer the random person on the street.

The easiest way to get a concrete view on any of this would be to define a resolution and see what homology and cohomology look like when you use this resolution to calculate them. Various different sources like to use various different resolutions. The one presented below is often called the bar resolution. [2]

Definition 3.2. The bar resolution is a free resolution of $\mathbb{Z}_{triv}$ denoted $\bar{C}_*$ with:

- $$\bar{C_n} := \bigoplus_{G^n} \mathbb{Z}[G]$$
  That is, the free $\mathbb{Z}[G]$-module on the set G^n. For each element $(g_1, \ldots, g_n) \in G^n$ we denote the corresponding generator in $\bar{C_n}$ by $[g_1, \ldots, g_n]$.
- The augmentation $\epsilon : \bar{C_0} = \mathbb{Z}[G] \to \mathbb{Z}$ is defined by:
  $$\epsilon\left(\sum a_g g\right) := \sum a_g$$
- The nth differential $\partial_n : \bar{C_n}(G) \to \bar{C_{n-1}}(G)$ is defined by:

$$\partial_n([g_1, \ldots, g_n]) := g_1[g_2, \ldots, g_n] + \sum_{k=1}^{n-1}(-1)^k[g_1, \ldots, g_k g_{k+1}, \ldots, g_n] + (-1)^n[g_1, \ldots, g_{n-1}]$$

A straightforward calculation shows that this is an exacts chain complex and thus a resolution.

3.2. **An explicit description of group cohomology.** To understand what cohomology represents you have to first recall what it means for a module to be free. Recall that for any free R-module $F = \bigoplus_{s\in S} R$ denoting for any $s \in S$ the corresponding basis element $[s]$. We have for any module A an isomorphism of abelian groups $j : \mathrm{Hom}_R(F, A) \to \mathrm{Hom}_{Set}(S, A)$ given by the assignment $j : f \mapsto f_j$ where $f_j(s) = f([s])$.

We are now able to give a more explicit description of cohomology groups. Given a group G and a G-module A, the cohomology cochain complex, $\bar{C}^*(G, A)$ is given by taking $\bar{C^n}(G, A) := \mathrm{Hom}(\bar{C^n}(G), A)$ and differentials ∂_A^n to be induced by the Hom functor. Then it is immediate from the definition of cohomology that taking the cohomology of this cochain complex. By the above discussion we may view $\bar{C^n}(G, A)$ as the abelian group of set functions from $G^n \to A$. Finally giving an explicit construction of ∂_A^n we get:

$$\begin{aligned}(\partial_A^n f)(g_1, \ldots, g_{n+1}) &:= (f \circ \partial_{n+1})(g_1, \ldots, g_{n+1}) \\ &= g_1 f(g_2, \ldots, g_{n+1}) \\ &+ \sum_{k=1}^{n}(-1)^k f(g_1, \ldots, g_k g_{k+1}, \ldots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \ldots, g_n)\end{aligned}$$

All of this to say that the nth cohomology group is explicitly the set of all functions $G^n \to A$ such that:

$$g_1 f(g_2, \ldots, g_{n+1}) = \sum_{k=1}^{n}(-1)^{k+1} f(g_1, \ldots, g_k g_{k+1}, \ldots, g_{n+1}) + (-1)^n f(g_1, \ldots, g_n)$$

Mod functions of the form:

$$g_1 f(g_2, \ldots, g_n) + \sum_{k=1}^{n-1}(-1)^k f(g_1, \ldots, g_k g_{k+1}, \ldots, g_n) + (-1)^n f(g_1, \ldots, g_{n-1})$$

We'll get into some specific applications soon, but for right now a good picture to have in your head is that group cohomology gives us an idea of the external structure of a given G-module. It tells us how the structure of a given G-module is "observed" by the group G.

3.3. **An explicit description of Homology.** If cohomology tells us about the external structure of a G-module with respect to the group G, the homology should tell us about the internal structure with respect to G.

Again we start with a given G-module A and the bar resolution. This time though, instead of taking $\mathrm{Hom}(\bar{C}_n(G), A)$ we take the tensor product. That is, the nth homology chain complex $\bar{C}_n(G, A) := \bar{C}_n \otimes_{\mathbb{Z}[G]} A$ by 2.2 we thus have that $\bar{C}_n(G, A) = A^{G^n}$. The maps $\partial_n^A : A^{G^n} \to A^{G^{n-1}}$ are induced by the tensor product, given explicitly:

$$\begin{aligned}\partial_n^A([g_1, \ldots, g_n] \otimes a) = [g_2, \ldots, g_n] \otimes g_1 a \\ + \sum_{j=1}^{n-1} (-1)^j ([g_1, \ldots, g_j g_{j+1}, \ldots, g_n] \otimes a) \\ + (-1)^n ([g_1, \ldots, g_{n-1}] \otimes a)\end{aligned}$$

Then the nth homology group is given explicitly as the cycles $Z_n(G, A)$. That is, the subgroup of A^{G^n} generated by the elements $[g_1, \ldots, g_n]$ satisfying:

$$[g_2, \ldots, g_n] \otimes g_1 a = \sum_{j=1}^{n-1} (-1)^{j-1} ([g_1, \ldots, g_j g_{j+1}, \ldots, g_n] \otimes a) + (-1)^{n+1} ([g_1, \ldots, g_{n-1}] \otimes a)$$

Modulo the Boundaries $B_n(G, A)$; elements of the form:

$$\begin{aligned}&[g_2, \ldots, g_{n+1}] \otimes g_1 a \\ &+ \sum_{j=1}^{n} (-1)^j ([g_1, \ldots, g_j g_{j+1}, \ldots, g_{n+1}] \otimes a) \\ &+ (-1)^n ([g_1, \ldots, g_n] \otimes a)\end{aligned}$$

So where cohomology groups are quotients of subroups of functions into a space, homology groups are subquotients of a space. You should think of homology as capturing the internal structure of a G-module.

4. Low dimensional (Co)Homology

4.1. **Crossed homomorphisms and H^1.** The subgroup H_1 admits a very simple description.

Proposition 4.1. *Let G be a group and A a G-module. Then $H^1(G, A)$ is isomorphic to the group of functions $f : G \to A$ such that $f(gh) = f(g) + gf(h)$ modulo functions f such that for some fixed $a \in A$ and all $g \in G$ $f(g) = ga - g$.*

Proof. We have already given a description of ∂_A^n. Since

$$(\partial_A^1 f)(g, h) = gf(h) - f(gh) + f(g)$$

we see that $Z^1(A) = \ker(\partial_A^1)$ consists of functions such that for all $g, h \in G$

$$f(g) - f(gh) + gf(h) = 0$$

In other words, those functions such that $f(g) + gf(h) = f(gh)$.

Similarly ∂_A^0 takes functions $f : \{*\} \to A$ to functions $\partial_A^0 f : G \to A$ by

$$\partial_A^0 f(g) = gf(*) - f(*)$$

since $f(*)$ is just some element $a \in A$ we thus have $\mathrm{Img}(\partial_A^0) = B^1$ consists of functions of the form $f(g) = ga - a$. Thus $H^1(G, A) = Z^1(A)/B^1(A)$. Which as we have just shown is exactly the group of functions $f : G \to A$ such that $f(gh) = f(g) + gf(h)$ mod out functions f such that for some fixed $a \in A$ and all $g \in G$ $f(g) = ga - g$. ■

In particular we have the following corollary.

Corollary 4.2. *Let G be a group and A a G-module such that for all $a \in A$ and $g \in G$ $ga = a$. Then $H^1(G, A) = \mathrm{Hom}(G, A)$.*

Proof. By the triviality of the action by G on A we have that $f(g)+gf(h) = f(g)+f(h)$ and thus that $\ker(\partial_A^1) = \mathrm{Hom}(G, A)$. Since $ga - a = a - a = 0$ it is immediate that $\mathrm{Img}(\partial_A^0) = 0$. Thus $H^1(G, A) = \ker(\partial_A^1)/\mathrm{Img}(\partial_A^0) = \ker(\partial_A^1)/\{0\} = \ker(\partial_A^1) = \mathrm{Hom}(G, A)$. ■

The functions in $\ker(\partial_A^n)$ are called crossed homomorphisms and they come up very often whenever you have a group and a G-module.

Example. One of the most interesting applications of H^1 is to finding outer automorphisms of an affine group.

Definition 4.3. Let R be a commutative ring. Then the affine group $\mathrm{Aff}(R)$ consists of functions $f : R \to R$ of the form $f(x) = ux + r$ where $u, r \in R$ with u invertible in R. The group operation is given by composition of functions. This is easily checked to be a group:

- Associativity is given by definition of function composition.
- f(x)=x is the identity since $uf(x) + r = ux + r = f(ux + r)$.
- If $f(x) = ux + r$ define $f^{-1}(x) := u^{-1}x - u^{-1}r$, then

$$f(f^{-1}(x)) = u(u^{-1}x - u^{-1}r) + r = x - r + r = x$$

  Thus every element has an inverse.

Whenever we have an interesting group we want to know more about it is often helpful to look at its automorphism group. Every group has what are called inner automorphisms. These are automorphisms of the form $\gamma_g(x) = gxg^{-1}$ for some fixed element g of the group. While inner automorphisms are interesting in their own right, they can only really tell us about the conjugacy action of a group. If we want to know anything more about a group we have to look at those automorphisms that are not inner. With that in mind we have the following definition.

Definition 4.4. The outer automorphism group of a group G is defined to be the quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ where $\mathrm{Inn}(G)$ denotes the subgroup of inner automorphisms.

Additionally those automorphisms φ such that if $f(x) := ux + r$ $\varphi(f) = g$ such that $g(x) = ux + r + D(g)$. Notably such automorphisms define a subgroup of the automorphism group which we will denote $\mathrm{Aut}(\mathrm{Aff}(R); R)$. We thus have the following definition.

Definition 4.5. Let $\mathrm{Inn}(\mathrm{Aff}(R); R)$ denote the intersection of $\mathrm{Aut}(\mathrm{Aff}(R); R)$ with $\mathrm{Inn}(\mathrm{Aff}(R))$. Then we define $\mathrm{Out}(\mathrm{Aff}(R); R) := \mathrm{Aut}(\mathrm{Aff}(R); R)/\mathrm{Inn}(\mathrm{Aff}(R); R)$.

Now note that R is actually an $R^\times$-module under the multiplication action. We now show that $\mathrm{Out}(\mathrm{Aff}(R); R) \cong H^1(R^\times, R)$.

Proposition 4.6. $\mathrm{Aut}(\mathrm{Aff}(R); R) \cong Z^1(R^\times, R)$

Proof. First suppose $\varphi \in \text{Aut}(\text{Aff}(R);R)$ then $\varphi(ux+r) = ux+r+D(u)$ where $D(u)$ is some set function. Then in fact $D(u) \in Z^1(R^\times, R)$. This is easily shown because:

$$\begin{aligned}\varphi(uvx+0) &= uvx + D(uv)\\ &= \varphi(ux+0 \circ vx+0)\\ &= \varphi(ux+0)\circ\varphi(vx+0)\\ &= u(vx+D(v)) + D(u)\\ &= uvx + uD(v) + D(u).\end{aligned}$$

Therefore $D(uv) = D(u) + uD(v)$ and thus $D \in Z^1(R^\times, R)$.

Similarly if D is any cocycle. Then define $\varphi(ux+r) := ux+r+D(u)$. This has a clear inverse given by $\varphi^{-1}(ux+r) := ux+r-D(u)$ and is a homomorphism because:

$$\begin{aligned}\varphi(ux+r\circ vx+s) &= \varphi(uvx+us+r)\\ &= uvx+us+r+D(uv)\\ &= uvx+us+r+uD(v)+D(u)\\ &= u(vx+s+D(v))+r+D(u)\\ &= \varphi(ux+r)\circ\varphi(vx+s)\end{aligned}$$

$\text{Aut}(\text{Aff}(R);R) \cong Z^1(R^\times, R)$ ■

Proposition 4.7. $B^1(R^\times, R) \cong \text{Inn}(\text{Aff}(R);R)$

Proof. Let $f = vx+s$ be such that its inner automorphism $\gamma_f \in \text{Inn}(\text{Aff}(R);R)$. Then:

$$\begin{aligned}\gamma_f(ux+r) &= v(u(v^{-1}x - v^{-1}s)+r)+s\\ &= vuv^{-1}x - vuv^{-1}s + vr + s\\ &= ux + s - us + vr\\ &= ux + r + D(u)\end{aligned}$$

Thus f must be of the form $x+s$ and $D(u) = s - su \in B^1(R^\times, R)$.

Now let $D(u) \in B^1(R^\times, R)$, then $D(u) = s - su$ for some fixed $s \in R$. Then we have just shown that if $f = x+s$ $\gamma_f(ux+r) = ux+r+s-su$. Thus $B^1(R^\times, R) \cong \text{Inn}(\text{Aff}(R);R)$ ■

Theorem 4.8. $H^1(R^\times, R) \cong \text{Out}(\text{Aff}(R);R)$

Proof. Since $B^1(R^\times, R) \cong \text{Inn}(\text{Aff}(R);R)$ and $Z^1(R^\times, R) \cong \text{Aut}(\text{Aff}(R);R)$ we thus have $H^1(R^\times, R) = Z^1(R^\times, R)/B^1(R^\times, R) \cong \text{Aut}(\text{Aff}(R);R)/\text{Inn}(\text{Aff}(R);R) \cong \text{Out}(\text{Aff}(R);R)$. ■

4.2. **Extensions and H^2.** The second cohomology group H^2 is extremely useful. This is precisely because it classifies what are called extensions.

Definition 4.9. Given a group G and a group H. An extension of G by H is a short exact sequence:

$$1 \longrightarrow H \longrightarrow X \longrightarrow G \longrightarrow 1$$

So the group X has normal subgroup isomorphic to H with the quotient $X/H \cong G$.

Extensions give all ways to combine two groups. The reader should already be familiar with a couple of examples of extensions.

Example (Product)*.* The product of two groups G and H is an example of an extension. In fact it is in many ways the trivial extension. To be explicit the injection $H \to G \times H$ is given by $h \mapsto (1, h)$ and the surjection $G \times H$ is given by $(g, h) \mapsto g$.

Another example that we have just discussed is the affine group.

Example. The affine group of a given commutative ring R is an extension of the multiplicative group of R, $R^\times$ by its additive group R_A. The injection is given by $r \mapsto x + r$ and the surjection is given by $ax + r \mapsto a$.

Both of the above are examples of semidirect products. All this really means is that it's an extension $1 \to H \to X \to G \to 1$ where G is a subgroup of X. Importantly, most extensions are not semidirect products. We also only ever consider extensions up to equivalence.

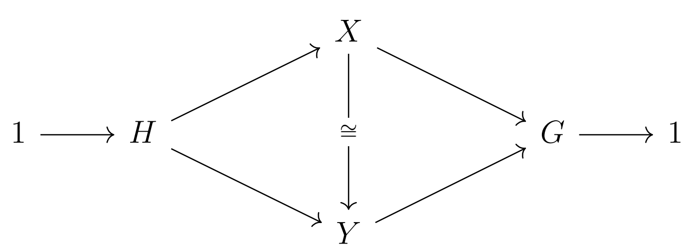
Definition 4.10 (equivalence of extensions)**.** Two extensions

$$1 \longrightarrow H \longrightarrow X \longrightarrow G \longrightarrow 1$$

and

$$1 \longrightarrow H \longrightarrow Y \longrightarrow G \longrightarrow 1$$

are said to be equivalent if we have an isomorphism $X \to Y$ such that



Commutes.

H^2 allows us to classify extensions of abelian kernel, that is extensions of a group G by an abelian group A. What makes this classification possible is the following lemma.

Lemma 4.11. *Let G be a group and A an abelian group. Then for any extension*

$$1 \longrightarrow A \xrightarrow{j} X \xrightarrow{\pi} G \longrightarrow 1$$

we have an induced action by G on A.

Proof. Let

$$1 \longrightarrow A \xrightarrow{j} X \xrightarrow{\pi} G \longrightarrow 1$$

Be some extension of G by A. Suppose $g \in G$ and $x, y \in X$ are such that $\pi(x) = \pi(y) = g$ then we have some $b \in A \hookrightarrow X$ such that $j(b) = xy^{-1}$. Thus for any $a \in A$ we have $j(a)xy^{-1} = xy^{-1}j(a)$ which implies $x^{-1}j(a)x = y^{-1}j(a)y$ and thus $xj(a)x^{-1} = yj(a)y^{-1}$. And thus conjugation by elements of G on elements of A in X is a well defined group action. We denote the application of this action on some a $\theta_g(a)$ ∎

Now the general goal is to fix a group G, an abelian group A and an action of G on A. That is we fix a group G and a G-module A. We then produce a bijection between $H^2(G, A)$ and the set of all extensions of G by A that induce the fixed action of G on A. This will however take several steps.

Lemma 4.12 (step 1)**.** *for any extension Let*

$$1 \longrightarrow A \longrightarrow X \xrightarrow{\pi} G \longrightarrow 1$$

we have a bijection between X and $A \times G$.

Proof. We first choose some section $r : G \to X$. This is a set function such that $\pi(r(g)) = g$. We can be assured that a section exists by the axiom of choice. We produce a bijection via the following function. $f : X \to A \times G$, $f(x) := (x \cdot r(\pi(x))^{-1}, \pi(x))$. Its fairly easy to show that this function is a bijection because it has an inverse $f^{-1}(a, g) := a \cdot r(g)$. This is its inverse since

$$f(f^{-1}(a, g)) = f(ar(g)) = (ar(g)(r(\pi(r(g)))^{-1}, \pi(ar(g))) = ((ar(g)(r(g))^{-1}, \pi(r(g))) = (a, g)$$

and

$$f^{-1}(f(x)) = f^{-1}(x(r(\pi(x))^{-1}, \pi(x)) = x(r(\pi(x))^{-1}r(\pi(x)) = x$$

■

Remark 4.13. This allows us to translate multiplication in X to multiplication in the set $A \times G$ by $f((f^{-1}(a, g))(f^{-1}(b, gh) = (\theta_g(b) + a + r(g)r(h)r(gh)^{-1}, gh)$

Lemma 4.14 (step 2)**.** *For any section r as described in Lemma 4.12 the function $R : G^2 \to A$ defined by $R(g, h) := r(g)r(h)r(gh)^{-1}$ is in $\ker(\partial_A^2)$.*

Proof. Recall that a function is in $\ker(\partial_A^2)$ if and only if it satisfies $g_1 f(g_2, g_3) = f(g_1g_2, g_3) - f(g_1, g_2g_3) + f(g_1, g_2)$. Since we have:

$$\begin{aligned}
\theta_{g_1}(R(g_2, g_3)) &= r(g_1)r(g_2)r(g_3)r(g_2g_3)^{-1}r(g_1)^{-1} \\
&= r(g_1)r(g_2)r(g_1g_2)^{-1}r(g_1g_2)r(g_3)r(g_1g_2g_3)r(g_1g_2g_3)^{-1}r(g_2g_3)^{-1}r(g_1)^{-1} \\
&= R(g_1, g_2) + R(g_1g_2, g_3) - R(g_1, g_2g_3)
\end{aligned}$$

R is thus in $\ker(\partial_A^2)$. ■

Remark 4.15. Lemma 4.14 gives us an assignment from extensions to $H^2(G, A)$ sending the extension to the class $[R]$. The below lemma shows that this assignment is independent of choice of section.

Lemma 4.16 (part 3)**.** *For any section r and any other section s with $S(g, h) := s(g)s(h)s(gh)^{-1}$ we have $S - R \in \mathrm{Img}(\partial_A^1)$.*

Proof. Because both s and r are sections of π which has kernel A $s(g) = ar(g)$ for some $a \in A$ and thus there is a set function $D : G \to A$ such that for all $g \in G$ $s(g) = D(g)r(g)$. Then

$$\begin{aligned}
S(g, h) - R(g, h) &= D(g)r(g)D(h)r(h)r(gh)^{-1}D(gh)^{-1} - r(g)r(h)r(gh)^{-1} \\
&= D(g)r(g)D(h)r(g)^{-1}r(g)r(h)r(gh)^{-1}D(gh)^{-1} - r(g)r(h)r(gh)^{-1} \\
&= D(g)\theta_g(D(h))r(g)r(h)r(gh)^{-1}D(gh)^{-1} - r(g)r(h)r(gh)^{-1} \\
&= D(g) + \theta_g(D(h)) + r(g)r(h)r(gh)^{-1} - D(gh) - r(g)r(h)r(gh)^{-1} \\
&= D(g) + \theta_g(D(h)) - D(gh)
\end{aligned}$$

Which is by definition in the image of ∂_A^1. ■

Lemma 4.17 (part 4)**.** *We have a well defined function assigning each class* $[R]$ *of* $H^2(G,A)$ *to the extension of* G *by* A*:*

$$1 \longrightarrow A \longrightarrow G_R \longrightarrow G \longrightarrow 1$$

where G_R *has underlying set* $A \times G$ *and multiplication defined by* $(a,g)(b,h) = (a + \theta_g(b) + R(g,h), gh)$

Proof. First we show that the operation described above defines a group. Associativity is proved as follows:

$$\begin{aligned}((a,g)(b,h))(c,f) &= (a+\theta_g(b)+R(g,h),gh)(c,f)\\ &= (a+\theta_g(b)+\theta_{gh}(c)+R(g,h)+R(gh,f),ghf)\\ &= (a+\theta_g(b)+\theta_{gh}(c)+\theta_g(R(h,f))+R(g,hf),ghf)\\ &= (a+\theta_g(b+\theta_h(c)+R(h,f))+R(g,hf),ghf)\\ &= (a,g)(b+\theta_h(c)+R(h,f)\\ &= (a,g)((b,h)(c,f))\end{aligned}$$

Now define $1_R := -R(1,1)$ then we have

$$\begin{aligned}(a,g)(1_R,1) &= (a+\theta_g(1_R)+R(g,1),g)\\ &= (a+\theta_g(-R(1,1))+R(g,1),g)\\ &= (a-R(g,1)+R(g,1)-R(g,1)+R(g,1),g)\\ &= (a,g)\end{aligned}$$

and

$$\begin{aligned}(1_R,1)(a,g) &= (1_R+a+R(1,g),g)\\ &= (a-R(1,1)+R(1,g)-R(1,g)+R(1,1),g)\\ &= (a,g)\end{aligned}$$

Thus $(1_R,1)$ is the identity element. Finally for any (a,g) we have

$$\begin{aligned}(a,g)(-\theta_{g^{-1}}(R(g,g^{-1})+R(1,1)+a),g^{-1}) &= (a+\theta_g(-\theta_{g^{-1}}(R(g,g^{-1})+R(1,1)+a))+R(g,g^{-1}),1)\\ &= (a-a-R(g,g^{-1})+R(g,g^{-1})-R(1,1),1)\\ &= (1_R,1)\end{aligned}$$

Therefore G_R is in fact a group. Now we have to show that this is an extension. We first define the injective homomorphism $i_R : A \to G_R$ by $i_R(a) = (a+1_R,1)$. This is easily checked to be a homomorphism by:

$$\begin{aligned}i_R(a)i_R(b) &= (a+1_R,1)(b+1_R,1)\\ &= (a+b+1_R+1_R+R(1,1),1)\\ &= (a+b+1_R,1)\\ &= i_R(a+b)\end{aligned}$$

and injectivity follows because $a \neq b$ implies $a+1_R \neq b+1_R$ which then implies $i_R(a) = (a+1_R,1) \neq (b+1_R,1) = i_R(b)$. The surjection $\pi : G_R \to G$ defined by $\pi(a,g) := g$. This is obviously surjective and it's a homomorphism because:

$$\pi((a,g)(b,h)) = \pi(a+\theta_g(b)+R(g,h),gh) = gh = \pi(a,g)\pi(b,h)$$

Since $\ker(\pi)$ consists of elements of the form $(a,1)$ which is exactly the image of i_R exactness follows.

Finally we have to show that if R and S are two elements of $\ker(\partial_A^2)$ which differ by an element D of $\mathrm{Img}(\partial_A^1)$ then they give equivalent extensions. So our assignment is well defined. We define $f : G_R \to G_S$ by $f(a,g) := (a + D(g), g)$. To show that $f(a,g)$ defines a homomorphism $G_R \to G_S$, we compute:

$$\begin{aligned} f((a,g)\cdot_R(b,h)) &= f(a+\theta_g(b)+R(g,h),gh) \\ &= (a+\theta_g(b)+R(g,h)+D(gh),gh) \end{aligned}$$

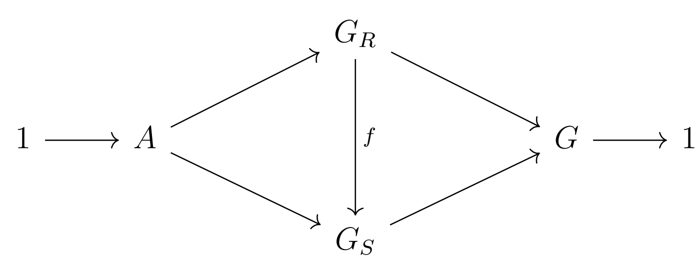
On the other hand:

$$\begin{aligned} f(a,g)\cdot f(b,h) &= (a+D(g),g)\cdot(b+D(h),h) \\ &= (a+D(g)+\theta_g(b+D(h))+S(g,h),gh) \\ &= (a+D(g)+\theta_g(b)+\theta_g(D(h))+S(g,h),gh) \end{aligned}$$

Now using the assumption that $R(g,h) = S(g,h)+D(g)+\theta_g(D(h))-D(gh)$, we substitute:

$$\begin{aligned} f(a,g)\cdot f(b,h) &= (a+\theta_g(b)+R(g,h)+D(gh),gh) \\ &= f((a,g)\cdot(b,h)) \end{aligned}$$

Therefore, f is a group homomorphism. This is a equivalence because $f(i_R(a)) = f(a + 1_R, 1) = (a+1_R+D(1),1)$ Using the identity $R(1,1)-S(1,1) = D(1)+D(1)-D(1)$ We thus get that $(a+1_R+D(1),1) = (a+1_S,1) = i_S(a)$ and $\pi(f(a,g)) = \pi(a+D(g),g) = g = \pi(a,g)$ meaning that



commutes ■

Theorem 4.18. *We have a one to one correspondence between elements of $H^2(G,A)$ and extensions with induced action corresponding to the G-module action on A.*

Proof. By remark 4.15 the assignment described by lemma 4.14 and 4.16 and the assignment described by lemma 4.17 are mutually inverse and thus bijections. ■

5. A few structural tools

Cohomology is particularly useful because of how structured it is. If cohomology shows up anywhere you immediately get a bunch of free structure. For example, you get a nice long exact sequence.

5.1. Long exact sequences and Tate cohomology.

Proposition 5.1 (Long exact sequence of cohomology)**.** *If* $0 \to A \to B \to C \to 0$ *is a short exact sequence of G-modules we have an induced long exact sequence:*

$$\dots \longrightarrow H^i(G,A) \longrightarrow H^i(G,B) \longrightarrow H^i(G,C) \longrightarrow H^{i+1}(G,A) \longrightarrow \dots$$

Proof. First note that $C^i(G,-) = \mathrm{Hom}(\mathbb{Z}[G],-)$. Since $\mathbb{Z}[G]$ is free we therefore have that if

$$0 \to A \to B \to C \to 0$$

is exact, so is

$$0 \to C^i(G,A) \to C^i(G,B) \to C^i(G,C) \to 0.$$

Thus

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^i(G,A) & \longrightarrow & C^i(G,B) & \longrightarrow & C^i(G,C) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial^i_A} & & \downarrow{\scriptstyle \partial^i_B} & & \downarrow{\scriptstyle \partial^i_C} & & \\ 0 & \longrightarrow & C^{i+1}(G,A) & \longrightarrow & C^{i+1}(G,B) & \longrightarrow & C^{i+1}(G,C) & \longrightarrow & 0 \end{array}$$

commutes and has exact rows. Then because $\bar{C}^*$ is a cochain complex we thus have that this diagram

$$\begin{array}{ccccccccc} & & C^i(G,A)/B^i(G,A) & \longrightarrow & C^i(G,B)/B^i(G,B) & \longrightarrow & C^i(G,C)/B^i(G,C) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial^i_A} & & \downarrow{\scriptstyle \partial^i_B} & & \downarrow{\scriptstyle \partial^i_C} & & \\ 0 & \longrightarrow & Z^{i+1}(G,A) & \longrightarrow & Z^{i+1}(G,B) & \longrightarrow & Z^{i+1}(G,C) & & \end{array}$$

commutes and has exact rows. Applying the snake lemma we then get an exact sequence

$$H^i(G,A) \longrightarrow H^i(G,B) \longrightarrow H^i(G,C) \longrightarrow H^{i+1}(G,A) \longrightarrow H^{i+1}(G,B) \longrightarrow H^{i+1}(G,C)$$

■

We have an identical theorem in the case of group homology. The proof is nearly identical and may be obtained by replacing every instance of C^i, C^{i+1}, Z^{i+1}, B^i, H^i, H^{i+1}, and $\mathrm{Hom}(\mathbb{Z}[G],-)$ in the above proof with C_i, C_{i+1}, Z_{i+1}, B_i, H_i, H_{i+1}, and $\mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} -$ respectively.

When G is finite we are able to extend the exact sequences to construct something called Tate cohomology. First, however, we have to define the norm element.

Definition 5.2. The norm element $N_G \in \mathbb{Z}[G]$ is defined to be the sum:

$$N_G = \sum_{g \in G} g$$

Importantly the norm element and its multiples are the only elements of $\mathbb{Z}[G]^G$. This means that left multiplication by the norm element induces a map from $A_G \to A^G$.

Lemma 5.3. *Let G be a finite group and A be a G-module. Then the left multiplication map by N_G on A induces a map $A_G \to A^G$.*

Proof. Since $A_G = A/\mathrm{Span}\{ag - a | a \in A, g \in G\}$ we need to show that $N_G(ag - a) = 0$. This follows because:

$$N_G(ag - a) = N_G(g - 1)(a) = (N_G - N_G)a = 0$$

Thus multiplication by N_G gives a well defined map $A_G \to A$. Now note that if $a \in A$ lies in the image of this map we must have $a = N_G b$ for some $b \in A$. Thus for any $g \in G$ we have $ga = gN_G b = N_G b = a$. Therefore $a \in A^G$ meaning multiplication by N_G gives a well defined map $\bar{N}_G : A_G \to A^G$ ■

Definition 5.4. We define $\hat{H}^0(G, A) = \mathrm{Coker}(\bar{N}_G)$ and $\hat{H}^{-1}(G, A) = \ker(\bar{N}_G)$.

We are now able to define Tate cohomology.

Definition 5.5. We have already defined Tate cohomology for $i = -1, 0$ so now we define for any $i > 0$, $\hat{H}^i(G, A) := H^i(G, A)$ and for $i < -1$ we define $\hat{H}^i(G, A) := H_{-i-1}(G, A)$.

The primary reason Tate cohomology exists at all is because it allows us to combine the homology and cohomology long exact sequences.

Lemma 5.6. *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence. Then we have a long exact sequence

$$\dots \longrightarrow \hat{H}^i(G, A) \longrightarrow \hat{H}^i(G, B) \longrightarrow \hat{H}^i(G, C) \longrightarrow \hat{H}^{i+1}(G, A) \longrightarrow \dots$$

Proof. Apply the snake lemma to this diagram

$$\begin{array}{ccccccccc} \dots \longrightarrow H_1(G,A) & \longrightarrow & H_0(G,A) = A_G & \longrightarrow & H_0(G,B) = B_G & \longrightarrow & H_0(G,C) = A_C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bar{N}_G} & & \downarrow{\scriptstyle \bar{N}_G} & & \downarrow{\scriptstyle \bar{N}_G} & & \\ 0 & \longrightarrow & H^0(G,A) = A^G & \longrightarrow & H^0(G,B) = B^G & \longrightarrow & H^0(G,C) = C^G & \longrightarrow & H^1(G,A) \longrightarrow \dots \end{array}$$

■

When G is not only finite but cyclic Tate cohomology further simplifies to become periodic. This comes from the following resolution.

Definition 5.7. Let $\mathbb{Z}/n$ be a cyclic group and denote its generator g. Then the following periodic chain complex is a resolution of $\mathbb{Z}_{triv}$:

$$\dots \longrightarrow \mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G] \xrightarrow{N_G} \mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z}_{triv} \longrightarrow 0$$

where the map $g - 1$ denotes multiplication by $g - 1$ and N_G denotes multiplication by N_G.

Theorem 5.8. *Tate cohomology of cyclic groups is periodic*

Proof. If we use the above chain complex to compute Tate cohomology $\hat{H}^i(G, A)$, noting that $\mathrm{Hom}(\mathbb{Z}[G], A) = A = \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A$ we see that:

- The chain complex used to compute typical group cohomology is

$$0 \longrightarrow A \xrightarrow{g-1} A \xrightarrow{N_G} A \xrightarrow{g-1} A \xrightarrow{N_G} A \longrightarrow \dots$$

- The chain complex used to compute typical group homology is

$$\dots \longrightarrow A \xrightarrow{g-1} A \xrightarrow{N_G} A \xrightarrow{g-1} A \xrightarrow{N_G} A \longrightarrow 0$$

- Because the homomorphism connecting group homology and cohomology is multiplication by N_G, the sequence used to compute Tate cohomology is

$$\dots \longrightarrow A \xrightarrow{g-1} A \xrightarrow{N_G} A \xrightarrow{g-1} A \xrightarrow{N_G} A \longrightarrow \dots$$

Thus because the chain complex used to compute Tate cohomology is periodic. So is Tate cohomology. ■

5.2. **Compatible maps.** We now talk a bit about compatible maps. These allow us to relate cohomology groups $H^i(G,A)$ and $H^i(K,B)$ by relating the groups G and K and relting the modules A and B.

Definition 5.9. Let G and H be groups, A a G-module and B an H-module. We say that a pair of maps (α,β), $\alpha : H \to G$, $\beta : A \to B$ are compatible if

$$\beta(\alpha(h)a) = h\beta(a)$$

Compatible maps provide a lot of utility because of the following lemma.

Lemma 5.10. *Suppose we have compatible maps $\alpha : H \to G$ and $\beta : A \to B$. Then the maps $\theta^i : C^i(G,A) \to C^i(H,B)$ given by:*

$$(\theta^i f)(h_1,\dots,h_i) := \beta(f(\alpha(h_1),\dots \alpha(h_i))$$

induce maps on the homology groups.

Proof. It is sufficient to show that θ^i sends coboundaries to coboundaries and cocycles to cocycles. In other words we need to show that:

$$(\partial_B^i \theta^i(f))(h_1,\dots,h_{i+1}) = (\theta^{i+1}\partial_A^i f)(h_1,\dots,h_{i+1})$$

This is a fairly simple calculation:

$$\begin{aligned}
(\theta^{i+1}\partial_A^i f)(h_1,\dots,h_{i+1}) &= \beta((\partial_A^i f)(\alpha(h_1),\dots,\alpha(h_{i+1}))) \\
&= \beta(\alpha(h_1)f(\alpha(h_2),\dots,\alpha(h_{n+1}))) \\
&+ \sum_{k=1}^{n}(-1)^k\beta(f(\alpha(h_1),\dots,\alpha(h_k)\alpha(h_{k+1}),\dots,\alpha(h_{n+1}))) \\
&+ (-1)^{n+1}\beta(f(\alpha(h_1),\dots,\alpha(h_n))) \\
&= h_1\beta(f(\alpha(h_2),\dots,\alpha(h_{n+1}))) \\
&+ \sum_{k=1}^{n}(-1)^k\beta(f(\alpha(h_1),\dots,\alpha(h_k)\alpha(h_{k+1}),\dots,\alpha(h_{n+1}))) \\
&+ (-1)^{n+1}\beta(f(\alpha(h_1),\dots,\alpha(h_n))) \\
&= (\partial_B^i\theta^i(f))(h_1,\dots,h_{i+1})
\end{aligned}$$

■

There are three canonical applications of compatible maps. The first and simplest of which is restriction:

Definition 5.11. Let $K \subset G$ be a subgroup of G and $i : K \hookrightarrow G$ the inclusion of K into G. Let A be a G module (and thus also a K-module). Then clearly i and id_A are compatible. We define restriction to be the map

$$\text{Res} : H^n(G, A) \to H^n(K, A)$$

induced by i and id_A.

When $K \subset G$ is a subgroup we have a very nice relation between their cohomologies. In order to see this we need to introduce the concept of induced and coinduced modules.

Definition 5.12. Let G be a group and K a subgroup of G. Then We can easily make any $\mathbb{Z}[G]$-module to a $\mathbb{Z}[K]$-module simply by restricting scalars. However if we want to turn a $\mathbb{Z}[K]$-module M to a $\mathbb{Z}[G]$ module we have to ways. The coinduced module is defined as $\text{CoInd}(M)_K^G := \text{Hom}_{\mathbb{Z}[K]}(\mathbb{Z}[G], M)$ and given module structure by $(gf)(x) := f(xg)$. The induced module is defined as $\text{Ind}_K^G(M) := \mathbb{Z}[G] \otimes_{\mathbb{Z}[K]} M$ and given module structure by $g(h \otimes m) := gh \otimes m$.

Coinduced modules allow us to introduce a particular pair of compatible maps.

Proposition 5.13. *let G be a group and K a subgroup of G. Let A be a K-module. Then the inclusion map $i : K \hookrightarrow G$ is compatible with the map $\phi : \text{CoInd}_K^G(A) \to A$ given by $\phi(f) := f(1)$.*

Proof. Let $f \in \text{CoInd}(A)$ and $k \in K$. Then

$$\phi(i(k)f) = \phi(kf) = f(k) = kf(1) = k\phi(f)$$

Meaning i and ϕ are compatible. ■

Lemma 5.14 (Shapiro's lemma)**.** *Let G be a group, K a subgroup of G and A a K-module. Then $H^n(G, \text{CoInd}_K^G(A)) \cong H^n(K, A)$.*

Proof. It is sufficent to show that the map $\theta^i : C^i(G, \text{CoInd}_K^G(A)) \to C^i(K, A)$ given by $(\theta^i f)(k_1, \ldots, k_i) := \phi(f(k_1, \ldots, k_i))$ is a bijection. First suppose $f \in \ker(\theta^i)$ then for all $k_1, \ldots, k_i \in K$ and $g \in G$ we have:

$$(f(k_1, \ldots, k_i))(g) = \phi(g(f(k_1, \ldots, k_i)) = \phi(f(gk_1, \ldots, gk_i)) = 0$$

And thus $f = 0$. Therefore θ^i is injective.

Now, if $\alpha \in C^i(K, A)$ define $f(k_1, \ldots, k_i)(g) := \alpha(gk_1, \ldots, gk_i)$, then by definiton $\theta^i(f) = \alpha$. Meaning θ^i is surjective. Thus $H^n(G, \text{CoInd}_K^G(A)) \cong H^n(K, A)$. ■

We now have an easy way of relating the cohomology groups of a group G and any of its subgroups K. But what if instead of being a subgroup, K is a quotient.

Lemma 5.15. *Let G be a group and N a normal subgroup of G and A be a G-module. Then let $\pi : G \to G/N$ be the projection map of G onto G/N and $i : A^N \hookrightarrow A$ the inclusion of A^N into A. Then π and i are compatible.*

Proof. Note that for any $a \in A^N$ and any $h, g \in G$ with $hN = gN$, then $h^{-1}g \in N$, thus $h^{-1}ga = a$ implying $ga = ha$. We therefore have that $i(\pi(g)a) = \pi(g)a = ga = gi(a)$

Thus π and i are compatible. ■

Definition 5.16. We define inflation to be the map $H^i(G/N, A^N) \to H^i(G, A)$ induced by i and π and denote it by Inf.

Remark 5.17. It is hopefully clear that inflation is injective. If two functions $f, g : G/N \to A^N$ are inequivalent then, if $\pi : G \to G/N$ is the natural surjection, $f \circ \pi \neq g \circ \pi$.

Theorem 5.18. *For any group G normal subgroup N and G-module A we have an exact sequence:*

$$0 \longrightarrow H^1(G/N, A^N) \xrightarrow{\text{Inf}} H^1(G, A) \xrightarrow{\text{Res}} H^1(N, A)$$

Proof. We have already discussed the injectivity of Inf, but we also have that if $f \in \text{Img}(\text{Inf})$ and $n \in N$ then

$$f(n) = f(\pi(n)) = f(\pi(1)) = f(1)$$

. Since

$$f(1) = f(1) + 1f(1) = 2f(1)$$

implies $f(1) = 0$, we therefore have

$$f(n) = f(1) = 0$$

meaning $\text{Res}(f) = 0$. Thus $\text{Img}(\text{Inf}) \subset \ker(\text{Res})$. Now suppose $f \in \ker(\text{Res})$. Then for all $n \in N$, $f(n) = 0$. Then if $g, h \in G$ are such that $gh^{-1} \in N$ we must have:

$$0 = f(h^{-1}g) = f(h^{-1}) + h^{-1}f(g) \tag{5.1}$$

meaning

$$-hf(h^{-1}) = f(g)$$

Since $0 = f(1) = f(h^{-1}) + h^{-1}f(h)$ we therefore have $-hf(h^{-1}) = f(h)$ meaning by equation 5.1

$$f(h) = f(g) \tag{5.2}$$

. Further, for any $g \in G$ and $n \in N$ we have

$$nf(g) = f(ng) - f(g) = f(ng)$$

which, since $ngg^{-1} = n \in N$ equation 5.2 gives that

$$nf(g) = f(ng) = f(g)$$

. This means that if $f \in \ker(\text{Res})$ then f induces a map $f^* : G/N \to A^N$ given by $f^*([g]) := f(g)$. Since clearly $\text{Inf}(f^*) = f$ we thus have that $\ker(\text{Res}) \subset \text{Img}(\text{Inf})$. Therefore $\ker(\text{Res}) = \text{Img}(\text{Inf})$ and thus

$$0 \longrightarrow H^1(G/N, A^N) \xrightarrow{\text{Inf}} H^1(G, A) \xrightarrow{\text{Res}} H^1(N, A)$$

is exact.

■

We also have an anologue of compatible maps for homology.

Definition 5.19. Let $\alpha : G \to H$ and $\beta : A \to B$. Then these maps are said to be homologically compatible if:

$$\beta(ga) = \alpha(g)\beta(a)$$

Lemma 5.20. *Compatible maps $\alpha : G \to H$ and $\beta : A \to B$ induce maps on homology $H_i(G, A) \to H_i(K, B)$.*

Proof. We have induced maps $C_i(G, A) \to C_i(K, B)$ given by $(\alpha_i \otimes \beta)([g_1, \dots g_i] \otimes a) := [\alpha(g_1), \dots \alpha(g_i)] \otimes \beta(a)$. Via a calculation that is nearly identical to the one given in the proof of lemma 5.10 these maps respect differentials and thus induce maps on homology. ■

We have analogous definitions and theorems for homologically compatible maps as we do compatbile maps. In particular we have

Definition 5.21. For a subgroup $H \subset G$ and G-module A corestriction is the map

$$H_i(G, A) \to H_i(K, A)$$

given by the homologically compatible maps $i : K \hookrightarrow G$ and $id_A : A \to A$.

Definition 5.22. If N is a normal subgroup of G and A is a G-module coinflation is the map:

$$H_i(G, A) \to H_i(G/N, A_N)$$

induced by the homologically compatible quotient maps $\pi : G \to G/N$ and $p : A \to A_N$.

We also have the following two lemmas.

Lemma 5.23 (Shapiro's lemma)**.** *If K is a subgroup of G and A is a K-module the $H_i(G, \mathrm{Ind}_K^G(A)) \cong H_i(K, A)$*

Lemma 5.24. *Suppose N is a normal subgroup of G and A is a G-module then we have an exact sequence:*

$$H_1(N, A) \xrightarrow{\text{Cor}} H_1(G, A) \xrightarrow{\text{CoInf}} H_1(G/N, A_H) \longrightarrow 0$$

Remark 5.25. The proofs of the above two lemmas closely parallel those of lemma 5.14 and theorem 5.18. To avoid unnecessary repetition, I have chosen not to write them out in full. Readers who wish to see the details are encouraged to adapt the earlier proofs accordingly.

6. Conclusion

Group cohomology provides a powerful framework for studying group actions on modules. We introduced homology and cohomology as derived functors, explored their low-dimensional interpretations (such as crossed homomorphisms and group extensions), and discussed key tools like long exact sequences and Tate cohomology. These concepts connect abstract homological algebra to concrete group-theoretic problems, making them essential for deeper investigations in mathematics.

We have notably excluded many applications of group cohomology to wider mathematics. If the reader would like to learn about galois cohomology, a detailed exposition may be found in [3]. similarly, [1] is a wonderful book that covers the theory of central simple algebras, which uses a lot of group cohomology. Finally if one wants a more geometric lens of group cohomology these notes [2] are simply amazing.

References

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